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D-brane effective action and tachyon condensation in topological minimal models

Manfred Herbst,[a] **Calin-Iuliu Lazaroiu**[b] **and Wolfgang Lerche**[a]

[a] *Department of Physics, CERN, Theory Division*
CH-1211 Geneva 23, Switzerland
[b] *5243 Chamberlin Hall, University of Wisconsin*
1150 University Ave, Madison, Wisconsin 53706, U.S.A.
E-mail: manfred.herbst@cern.ch, calin@physics.wisc.edu, wolfgang.lerche@cern.ch

Abstract: We study D-brane moduli spaces and tachyon condensation in B-type topological minimal models and their massive deformations. We show that any B-type brane is isomorphic with a direct sum of 'minimal' branes, and that its moduli space is stratified according to the type of such decompositions. Using the Landau-Ginzburg formulation, we propose a closed formula for the effective deformation potential, defined as the generating function of tree-level open string amplitudes in the presence of D-branes. This provides a direct link to the categorical description, and can be formulated in terms of holomorphic matrix models. We also check that the critical locus of this potential reproduces the D-branes' moduli space as expected from general considerations. Using these tools, we perform a detailed analysis of a few examples, for which we obtain a complete algebro-geometric description of moduli spaces and strata.

Keywords: Topological Field Theories, D-branes, Topological Strings.

Contents

1. Introduction

A central problem in $\mathcal{N}=1$ string compactifications with D-branes is the computation of superpotential terms arising from scattering of open strings. While this is far from being solved in nontrivial set-ups (for example D-branes on compact Calabi-Yau manifolds), one can gain important insights by studying more basic building blocks such as $N=2$ minimal models. It is by now well understood that the superpotential is computed by the associated topological string, and that it can be represented as the generating function of tree-level open string amplitudes (see [1, 2] for a detailed discussion of this aspect). This allows one to approach the subject with the powerful methods of topological string theory. Basic results in this respect were obtained in [2], where it was shown that the generating function of open string amplitudes (the so-called *effective potential* $\mathcal{W}_{\rm eff}$) satisfies a countable series of constraints which generalize the well-known associativity equations of [3], and that it can be viewed as a (non-cubic) string field action in an approximation in which the closed string variables are treated as backgrounds. While the interpretation of $\mathcal{W}_{\rm eff}$ as a space-time superpotential only makes sense in critical string theories, this quantity is well-defined for general models, and the properties mentioned above are not restricted to the critical case [2].

An important aspect of the effective potential is that it encodes all obstructions to D-brane deformations, namely the true D-brane moduli space can be represented locally as the critical set of $\mathcal{W}_{\rm eff}$ in a space of linearized deformations [1]. This connects obstruction theory to open string dynamics, and provides a tool for analyzing the local moduli space. In many examples, the latter is an algebraic variety and can be studied quite explicitly.

In the present paper, we study effective potentials and topological D-brane moduli spaces from a global perspective, focusing on B-twisted minimal models and their massive deformations.[1] Upon using the Landau-Ginzburg realization of such theories [7, 8], we propose a closed form for the effective potential, which expresses this quantity through the category-theoretic data of [9]–[14]. This generalizes results obtained by solving the consistency conditions of [2] in a series of examples, and should be viewed as a conjecture still awaiting proof. As a consistency check, we show that the (matrix) factorization locus of the Landau-Ginzburg superpotential coincides locally with the critical locus of $\mathcal{W}_{\rm eff}$. This concise proposal should be contrasted with the bulk WDVV (pre-)potential $\mathcal{F}$, for which a closed expression is not known. When combined with a general characterization of moduli spaces, it allows for a complete description of the topological version of tachyon condensation in such models.

Specifically, we shall consider minimal models of type A_{k+1}, whose target space in the Landau-Ginzburg realization is the affine line. In this case, the categorical description of B-type branes (originally proposed by Kontsevich [9] and further developed in [10]) is extremely simple. First, all projective modules over $\mathbb{C}[x]$ are free, which allows us to represent branes as pairs $\mathbb{M} = (M, D)$ where $M = M_+ \oplus M_-$ with $M_+ = M_- = \mathbb{C}[x]^{\oplus r}$ and D is a block off-diagonal polynomial matrix of type $(2r) \times (2r)$. Physically, M_+ and M_- represent r pairs of $D2$ branes and antibranes of the associated sigma model, which condense to configurations of $D0$-branes when turning on the Landau-Ginzburg superpotential [11]–[16]. Moreover, D models the boundary part of the BRST operator. Second, the ring of univariate polynomials is a PID (principal ideal domain), which implies that each $\mathbb{M}$ is isomorphic with a direct sum of pairs $\mathbb{M}_i = (M_i, D_i)$ for which $M_{+,i} = M_{-,i} = \mathbb{C}[x]$. Such 'rank one' objects are the analogs of 'rational D-branes' for massively perturbed B-type minimal models, and we shall call them 'minimal branes'.

Thus *every D-brane is isomorphic with a direct sum of minimal branes.* This observation implies that a D-brane's moduli correspond to varying the isomorphism between $\mathbb{M}$ and its minimal brane decomposition $\oplus_i \mathbb{M}_i$, and allows for a computable description of the boundary moduli space. In fact, we will show that such moduli spaces are stratified according to the minimal brane content. In particular, the minimal brane decomposition

[1]Some examples of boundary flows and tachyon condensation at the conformal point of the parameter space were previously discussed in [4, 5, 6].

changes each time one crosses from a stratum to another. Such transitions implement composite formation processes, thus giving an explicit description of 'topological tachyon condensation'[2] in minimal models and their massive deformations. This provides a set of theories for which open string tachyon condensation and D-brane composite formation can be studied exactly.

The paper is organized as follows. In section 2, we analyze general aspects of D-branes in minimal models and their massive deformations. We first recall some basic facts about D-branes in the Landau-Ginzburg formulation, focusing on the category-theoretical description of [9, 10]. After discussing the subcategory of minimal branes, we show that every D-brane is isomorphic with a direct sum of such objects, a result which relies on the existence of a Smith normal form for univariate polynomial matrices. We also determine the stratification of the moduli space of a D-brane induced by minimal brane decompositions. Section 3 considers the physical description of D-brane deformations induced by turning on the coupling to 'odd' boundary observables. After explaining the type of deformations considered in this paper, we discuss the parameterization of linearized deformations for minimal branes. We also show how composites obtained through topological tachyon condensation can be described as objects of the original D-brane category, and determine the appropriate parameterization of linearized deformations for such composites. In section 4, we discuss our proposal for the effective potential $\mathcal{W}_{\rm eff}$, starting with the case of minimal branes. After recalling the results obtained in [2], we cast them into a closed form which resembles a generalized residue formula. Subsection 4.2 extends this proposal to general B-type branes and shows that the critical set of the effective potential reproduces the deformation space. In subsection 4.3, we show that our proposal for $\mathcal{W}_{\rm eff}$ can be written as the classical potential of a holomorphic matrix model as defined and studied in [41], and explain how the 'constituent $D0$-branes' of a minimal brane arise in this description. Section 5 discusses moduli spaces and tachyon condensation processes in a series of examples. After recalling the physics-inspired parameterization introduced in section 3, we give a detailed discussion of moduli spaces for certain composites of minimal branes, which we describe completely as affine varieties. We also extract the associated strata, and discuss appropriate reparameterizations for the effective potentials along such strata.

2. D-branes in topological minimal models

In this section we discuss general aspects of D-branes in topological minimal models and their massive deformations. After recalling the description due to [9]–[16], we discuss the subcategory of minimal branes and show that any brane is isomorphic with a sum of minimal branes. We also show that the moduli space of a given D-brane is naturally stratified according to its minimal brane content.

[2] In the topological version of tachyon condensation, there are only topological (i.e. twisted) models for tachyons, and one deforms the theory along flat directions rather than starting from the top of a Mexican hat potential. Nevertheless, one encounters a truncation of the open string spectrum, and composite formation along such deformations. For critical topological string theories, this version of tachyon condensation was discussed abstractly in [17, 18] and analyzed in certain examples in [19]–[23] with the tools of string field theory (see [24] for a review).

2.1 Landau-Ginzburg description of B-type branes

Let us recall the construction of B-type branes in A_{k+1} topological minimal models. The bulk sector is described by a twisted $N = 2$ minimal model at SU(2) level k, together with its massive deformations. These theories have a convenient Landau-Ginzburg realization [7, 8, 3], in which the bulk sector is characterized by the superpotential:

$$W(x,t) = \frac{x^{k+2}}{k+2} - \sum_{i=0}^{k} g_{k+2-i}(t)\, x^i\,. \tag{2.1}$$

We will often write this polynomial in terms of its distinct roots $\tilde{x}_i(t)$:

$$W(x;t) = \frac{1}{k+2} \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{k_i}\,, \tag{2.2}$$

where $1 \leq s \leq k+2$ and $k_i \geq 1$ with $\sum_{i=1}^{s} k_i = k+2$. The polynomials $g_{k+2-i}(t)$ in (2.1) depend on flat coordinates t_i $(i = 2 \ldots k+2)$ as described in [3]. These coordinates measure the strength of couplings in the perturbed bulk action, namely:

$$\delta S = \sum_{i=0}^{k} t_{k+2-i} \int d^2z \{G_{-1}, [\bar{G}_{-1}, \Phi_i]\}\,, \tag{2.3}$$

where G_{-1} and $\bar{G}_{-1}$ are modes of the left and right moving twisted supercurrents. The case $t = 0$ (i.e., $W = \frac{x^{k+2}}{k+2}$) corresponds to the twisted $N = 2$ superconformal minimal model, while general values of t_i describe its massive topological deformations. The observables Φ_i form a linear basis of the space of chiral primary fields, which can be written as the Jacobi algebra of the Landau-Ginzburg superpotential:

$$\mathcal{R} := \mathbb{C}[x]/\langle \partial_x W^{(k+2)}(x) \rangle\,. \tag{2.4}$$

Thus one can choose the basis:

$$\Phi_i = x^i\,, \qquad i = 0 \ldots k\,, \tag{2.5}$$

which corresponds to the parameterization used in (2.1).

A general analysis of the boundary sector of B-twisted Landau-Ginzburg models was performed in [11]–[16], upon using a suggestion of [9].[3] The situation is simplest for single, 'minimal' branes (whose precise definition will be given below). Such branes correspond to all polynomial factorizations of the bulk superpotential:

$$W(x) = J(x)\, E(x)\,, \tag{2.6}$$

where $J(x)$ plays the role of a boundary superpotential [12]. As discussed in more detail in section 4.3, the zeroes of $J(x)$ can be viewed as locations of $D0$ branes in the target space $\mathbb{C}$. Moreover, E appears in the generalized chirality condition for fermionic boundary

[3] See [25]–[29] and [43] for previous work on the general subject of D-branes in Landau-Ginzburg models, and [30] for a review.

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superfields [29, 31]. The generalization to arbitrary D-branes is obtained by promoting J and E to square polynomial matrices (square matrices whose entries belong to the polynomial ring $\mathbb{C}[x]$). These describe [43] and [11]–[16] the tachyon profiles for a set of pairs of D2-branes and antibranes of the associated *sigma* model, which is recovered by turning off W.

In more abstract language, a general topological B-type brane is described by the pair of data $\mathbb{M} = (M, D)$, where M is a free $\mathbb{C}[x]$-supermodule and D an odd module endomorphism subject to the condition:[4]

$$D^2 = W \mathbb{1} . \tag{2.7}$$

The endomorphism D plays the role of a boundary BRST operator [13, 16], and can be viewed as a square polynomial matrix upon choosing an arbitrary basis of M. Considering the homogeneous decomposition $M = M_+ \oplus M_-$, which corresponds to a decomposition into sigma model $D2$ branes and antibranes, it is clear that (2.7) has no solutions unless $\mathrm{rk}\, M_+ = \mathrm{rk}\, M_-$, so we will always assume that this condition holds and denote the common rank by r. Using this decomposition, we write:

$$D = \begin{bmatrix} 0 & E \\ J & 0 \end{bmatrix} , \tag{2.8}$$

with $J \in \mathrm{Hom}_{\mathbb{C}[x]}(M_+, M_-)$ and $E \in \mathrm{Hom}_{\mathbb{C}[x]}(M_-, M_+)$. Expression (2.8) brings (2.7) to the form:

$$J\, E = E\, J = W\, \mathbb{1} , \tag{2.9}$$

which describes factorizations of the polynomial W into square polynomial matrices of type $r \times r$.

In this construction, the space of boundary topological observables is modeled by $\mathcal{D}$-cohomology, where $\mathcal{D}$ is the nilpotent operator on $\mathrm{End}_{\mathbb{C}[x]}(M)$ defined by taking the supercommutator of an endomorphism with D:

$$\mathcal{D} = [D, \cdot\,]. \tag{2.10}$$

With the product induced by composition of endomorphisms, the space $H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$ becomes a superalgebra over $\mathbb{C}$. Then the space of boundary observables can be identified with $H_{\mathbb{M}} := G \otimes_{\mathbb{C}} H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$, where G is a Grassmann algebra. An explicit analysis of $\mathcal{D}$-cohomology for minimal branes in minimal models and their massive deformations was performed in [12], and some of the results will be recalled below.

For general D-brane configurations, one also has boundary-condition changing observables which correspond to excitations of strings stretching between pairs of D-branes. Viewing these as morphisms between branes leads naturally to the category-theoretic picture expected from the work of [32]–[36] and [17, 18]. In Landau-Ginzburg models, the precise realization of this description was proposed by Kontsevich [9] and developed by

[4] A slightly more general construction is allowed [15, 16], but we shall not consider it here. This amounts to adding a constant operator to the right hand side of (2.7).

Orlov [10]. As expected from general considerations, the collection of all B-type branes forms an (enhanced) triangulated category DB_W, which in the present case is only $\mathbb{Z}_2$-graded. It arises as the cohomology category of a differential graded category DG_W, the so called 'Kontsevich category of pairs', which encodes off-shell tree-level open string data. By contrast, the cohomology category DB_W encodes the on-shell information.

The objects of DG_W are branes $\mathbb{M} = (M, D_M)$ as described above, viewed as pairs of $\mathbb{C}[x]$-module morphisms:

$$\mathbb{M} = \left(M_- \underset{J}{\overset{E}{\rightleftarrows}} M_+ \right) \tag{2.11}$$

subject to condition (2.9). Morphism spaces are defined by:

$$\mathrm{Hom}_{\mathrm{DG}_W}(\mathbb{M}, \mathbb{N}) = \bigoplus_{\alpha,\beta=+,-} \mathrm{Hom}(M_\alpha, N_\beta)\,, \tag{2.12}$$

with the obvious $\mathbb{Z}_2$ grading and the differential:

$$\mathcal{D}_{M,N}(f) = D_N \circ f - (-1)^{|f|} f \circ D_M\,, \tag{2.13}$$

where $|\cdot|$ denotes the degree of homogeneous morphisms. Boundary preserving observables in the sector $\mathbb{M}$ correspond to elements of $H_{\mathcal{D}_{M,M}}(\mathrm{Hom}(\mathbb{M}, \mathbb{M}))$, while boundary condition changing states are elements of $H_{\mathcal{D}_{M,N}}(\mathrm{Hom}(\mathbb{M}, \mathbb{N}))$ for $\mathbb{M} \neq \mathbb{N}$.

The 'on-shell' data is recovered as the total cohomology category $\mathrm{DB}_W := H(\mathrm{DG}_W)$, whose objects coincide with those of DG_W and whose morphism spaces are obtained by passing to cohomology in each $\mathrm{Hom}_{\mathrm{DG}_W}(\mathbb{M}, \mathbb{N})$. This category is triangulated[5] due to mathematical results of [40]. The construction is very similar in spirit to that of [17, 18] (see [24] for a review and [19]–[23] for applications to twisted sigma models). A different point of view, which does not rely directly on the results of [40], was discussed in [35, 36].

It is also useful to introduce an intermediate category P_W such that $Ob\mathrm{P}_W = Ob\mathrm{DG}_W = Ob\mathrm{DB}_W$ and $\mathrm{Hom}_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N}) = Z(\mathrm{Hom}_{\mathrm{DG}_W}(\mathbb{M}, \mathbb{N})) := \{f \in \mathrm{Hom}_{\mathrm{DG}_W}(\mathbb{M}, \mathbb{N}) | \mathcal{D}_{M,N}(f) = 0\}$, the space of cocycles in the complex $\mathrm{Hom}_{\mathrm{DG}_W}(\mathbb{M}, \mathbb{N})$. The latter has the decomposition $\mathrm{Hom}_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N}) = \mathrm{Hom}^0_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N}) \oplus \mathrm{Hom}^1_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N})$, where $\mathrm{Hom}^\alpha_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N}) = \{f \in \mathrm{Hom}_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N}) | f = \text{ homogeneous and } |f| = \alpha \in \mathbb{Z}_2\}$. Restricting morphism spaces to even components gives a subcategory P^0_W for which $\mathrm{Hom}_{\mathrm{P}^0_W}(\mathbb{M}, \mathbb{N}) = \mathrm{Hom}^0_{\mathrm{P}_W}(\mathbb{M}, \mathbb{N})$. Notice that two objects $\mathbb{M}, \mathbb{N}$ are isomorphic in P^0_W if and only if there exists an even module *isomorphism* $U \in \mathrm{Hom}^0_{\mathbb{C}[x]}(M, N)$ such that $D_N \circ U = U \circ D_M$; this is the natural notion of isomorphism between the branes $\mathbb{M}$ and $\mathbb{N}$.

Antibranes are described by acting with the shift functor [1], whose main effect is to flip J and E:

$$\mathbb{M}[1] = \left(M_+ \underset{-E}{\overset{-J}{\rightleftarrows}} M_- \right) \tag{2.14}$$

[5] More precisely, the subcategory $H^0(\mathrm{DG}_W)$ of DB_W is triangulated in the standard sense and $H^1(\mathrm{DG}_W) = H^0(\mathrm{DG}_W)[1]$ where [1] is the shift functor discussed below.

while acting in standard fashion on morphisms:

$$f = (f_0, f_1) \to f[1] = (f_1, f_0)\,. \tag{2.15}$$

Since the categories are only $\mathbb{Z}_2$-graded, we have $[1]^2 = \mathrm{id}$, where id is the identity functor.

2.2 The subcategory of minimal branes

The simplest class of objects in DG_W is obtained by choosing $\mathrm{rk}\, M_+ = \mathrm{rk}\, M_- = 1$, i.e. $M_+ = M_- = \mathbb{C}[x]$. Then J and E are polynomials satisfying (2.9), and they are easily described by using the factorization (2.2) of W. Namely

$$J(x) = C \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{m_i}\,, \qquad E(x) = \frac{1}{C(k+2)} \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{k_i - m_i}\,, \tag{2.16}$$

where C is a non-vanishing complex constant and the integers m_i satisfy $0 \le m_i \le k_i$. We let:

$$\ell + 1 := \deg J = \sum_{i=1}^{s} m_i\,. \tag{2.17}$$

When $C = 1$ (i.e. when J is a monic polynomial), each pair (2.16) is characterized by the integral vector $\mathbf{m} := (m_1 \dots m_s)$ and defines objects $\mathbb{M}_{\mathbf{m}}$ of DG_W which we shall call *minimal branes*. The collection of such objects defines a full subcategory of DG_W, the *minimal subcategory*. Notice that we keep the constant C explicitly, since we defined the objects of DG_W to be pairs (M, D) and not classes of such pairs up to a rescaling.[6] Moreover, notice that minimal branes are defined to have $C = 1$. The total number of minimal branes equals $\prod_{i=1}^{s}(k_i + 1)$.

The boundary preserving and boundary condition changing spectra of minimal branes were computed in [12]. The boundary preserving spectrum, described by $H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$, forms a super-commutative algebra $\mathbb{C}[x,\omega]/\mathcal{I}$ with even and odd generators x and ω. The ideal of relations can be described as follows. Let G denote the greatest common divisor of J and E, whose degree we denote by $l + 1$. We have:

$$J = p\,G \text{ and } E = q\,G \tag{2.18}$$

for some coprime polynomials p and q in the variable x. Then the odd generator is:[7]

$$\omega = \begin{bmatrix} 0 & q \\ -p & 0 \end{bmatrix}, \tag{2.19}$$

and the ideal $\mathcal{I}$ is generated by the elements:

$$G(x) \text{ and } \omega^2 + p(x)q(x)\,. \tag{2.20}$$

[6] Since we are in a quantum theory, rescaling by nonzero constants is not physically relevant (only the Hilbert space ray of any state is physically meaningful). One can implement this by passing to the category whose objects are equivalence classes of pairs (M, D) under such rescalings. Since this would clutter the notation, we prefer to use the original objects (M, D), with the understanding that such rescalings do not affect the underlying physics.

[7] The convention differs slightly from that of [12], and is fixed by agreement with (3.7) and (3.2).

To these generators we can associate rational degrees, which play the role of $U(1)$ charges at the superconformal point $t = 0$:

$$q(x) = 1 \,, \qquad q(\omega) = \frac{k}{2} - l \,. \tag{2.21}$$

Corresponding to the split $H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M)) = H^0_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M)) \oplus H^1_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$, one finds the homogeneous basis:

$$\{\phi_\alpha, \psi_\alpha\} = \{x^\alpha, \omega x^\alpha\} \,, \quad \alpha = 0 \dots l \,. \tag{2.22}$$

At the conformal point $t_i = 0$, we have $W = \frac{x^{k+2}}{k+2}$ and we can take $J = x^{\ell+1}$ and $E = \frac{x^{k+1-\ell}}{k+2}$. Then $G = J$, $p = 1$ and $q = \frac{x^{k-2\ell}}{k+2}$. This corresponds to $s = 1$, $k_1 = k+2$ and $l = \ell$, with $\ell \in \{-1 \dots k+1\}$. The number of minimal branes equals $k+3$, and they will be denoted by $\mathbb{M}_\ell \in Ob\mathrm{DG}_W$. As shown in [12], the branes $\mathbb{M}_\ell$ with $\ell = 0 \dots [k/2]$ correspond to the well-known 'rational' B-type boundary states of the A_{k+1} minimal model. The object $\mathbb{M}_{-1}$ is the trivial brane, which can be physically identified with the closed string vacuum.[8] The objects $\mathbb{M}_\ell$ with $\ell = [k/2] \dots k+1$ are the 'rational' antibranes, due to the relation $\mathbb{M}_\ell \approx \mathbb{M}_{k-\ell}[1]$ (for even k, the brane $\mathbb{M}_{k/2}$ is isomorphic with its antibrane). The choice $C = 1$ corresponds to a particular normalization of the 'rational' boundary states.

The boundary condition changing spectrum associated with open strings stretching between two minimal branes, is more complicated and we refer the reader to [12] for its general description. The simplest form is found at the conformal point $t_i = 0$, where $\mathrm{Hom}_{\mathrm{DB}_W}(\mathbb{M}_{\ell_1}, \mathbb{M}_{\ell_2})$ admits the following basis:

$$\left\{\phi^{\ell_1,\ell_2}_\gamma, \psi^{\ell_1,\ell_2}_\gamma\right\} = \left\{\beta^{(\ell_1,\ell_2)}x^\gamma, \omega^{(\ell_1,\ell_2)}x^\gamma\right\}, \quad \gamma = 0 \dots \ell_{12} := \min(\ell_1, \ell_2) \,. \tag{2.23}$$

Its even and odd generators:

$$\beta^{(\ell_1,\ell_2)} = \frac{1}{x^{\ell_{12}}}\begin{bmatrix} x^{\ell_2} & 0 \\ 0 & x^{\ell_1} \end{bmatrix}, \qquad \omega^{(\ell_1,\ell_2)} = \begin{bmatrix} 0 & x^{k-\ell_1-\ell_2} \\ 1 & 0 \end{bmatrix}, \tag{2.24}$$

have U(1) charges given by:

$$q(\beta^{(\ell_1,\ell_2)}) = \frac{1}{2}|\ell_1 - \ell_2| \,, \qquad q(\omega^{(\ell_1,\ell_2)}) = \frac{1}{2}(k - \ell_1 - \ell_2) \,. \tag{2.25}$$

2.3 Minimal brane decompositions

Returning to general D-branes, let us fix a $\mathbb{C}[x]$-supermodule $M = M_+ \oplus M_-$ with ranks given by $\mathrm{rk}\, M_+ = \mathrm{rk}\, M_- := r$. Then the factorization equation (2.7) is invariant under transformations of the form:

$$D \to UDU^{-1} \,, \tag{2.26}$$

where $U \in \mathrm{Aut}^0_{\mathbb{C}[x]}(M)$ is an even module automorphism of M. Writing $D = \left[\begin{smallmatrix} 0 & E \\ J & 0 \end{smallmatrix}\right]$ and:

$$U = S \oplus T = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}, \tag{2.27}$$

[8]More precisely, we have $\mathrm{Hom}_{\mathrm{DB}_W}(\mathbb{M}_{-1}, \mathbb{M}) = \mathrm{Hom}_{\mathrm{DB}_W}(\mathbb{M}, \mathbb{M}_{-1}) = 0$ for any D-brane M. Thus $\mathbb{M}_{-1}$ has trivial boundary preserving spectrum and trivial boundary changing spectrum with any other brane.

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with $S \in \mathrm{Aut}_{\mathbb{C}[x]}(M_+)$ and $T \in \mathrm{Aut}_{\mathbb{C}[x]}(M_-)$, we find that (2.26) amounts to the double similarity transformation:

$$\begin{aligned} E &\to SET^{-1} \\ J &\to TJS^{-1}\,. \end{aligned} \tag{2.28}$$

The operators S, T can be viewed as unimodular polynomial matrices, i.e. polynomial matrices which are invertible over $\mathbb{C}[x]$. This amounts to the requirement that their determinants are units in the polynomial ring, i.e. $\det S$ and $\det T$ are nonzero complex constants.

Since $\mathbb{C}[x]$ is a PID, the matrix J can always be brought to Smith form by a double similarity transformation. Namely, we have:

$$J = TJ_cS^{-1} \tag{2.29}$$

for some invertible T and S, with:

$$J_c = \mathrm{diag}(p_1 \ldots p_r)\,, \tag{2.30}$$

where $p_j \in \mathbb{C}[x]$ are monic univariate polynomials satisfying the division relations:

$$p_1|p_2|\ldots|p_r\,. \tag{2.31}$$

These are given explicitly by:

$$p_i := \frac{G_i}{G_{i-1}} \qquad \forall i = 1 \ldots r\,, \tag{2.32}$$

where G_i is the monic greatest common divisor of all $i \times i$ minors of J, with the convention $G_0 := 1$. It is clear that $\frac{W}{J}$ is a polynomial matrix if and only if $p_r|W$.

Hence a solution D of (2.7) can always be brought to the form:

$$D_c = \begin{bmatrix} 0 & E_c \\ J_c & 0 \end{bmatrix}, \tag{2.33}$$

where $J_c = \mathrm{diag}(p_1 \ldots p_r)$ and $E_c = \mathrm{diag}(q_1 \ldots q_r)$, with:

$$q_i := \frac{W}{p_i} \in \mathbb{C}[x]\,. \tag{2.34}$$

Thus $D = UD_cU^{-1}$ for some unimodular matrix U, where $D_c = D_1 \oplus \ldots \oplus D_r$, with:

$$D_i := \begin{bmatrix} 0 & q_i \\ p_i & 0 \end{bmatrix}. \tag{2.35}$$

This shows that any D-brane $\mathbb{M} = (M, D)$ is equivalent with the direct sum of minimal branes $\mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_r$, where $\mathbb{M}_i := (M_i, D_i) = (\mathbb{C}[x] \underset{p_i}{\overset{q_i}{\rightleftarrows}} \mathbb{C}[x])$, with $M_{+,i} = M_{-,i} := \mathbb{C}[x]$. The equivalence is implemented by the module isomorphism U^{-1}, which has the property

$U^{-1}D = (\oplus_i D_i)U^{-1}$ (showing that U^{-1} is an isomorphism in the category P^0_W). The direct sum $\oplus_i \mathbb{M}_i$ will be called the *minimal brane decomposition* of $\mathbb{M}$. We obtain the following:[9]

Proposition 1. *Any D-brane $\mathbb{M}$ is isomorphic in P^0_W with a direct sum of minimal branes. In particular, the finite subcategory of minimal branes generates DB_W as an additive category, up to isomorphisms.*

While the target space description of the isomorphism is elementary, we mention that the transformation $D \to UDU^{-1}$ amounts to a nontrivial change of variables in the world-sheet action. This is because the boundary coupling of [15] contains terms dependent on $D(\phi(\tau))$, where $\phi(\tau)$ is the restriction of the scalar field to the boundary of the world-sheet. The microscopic transformation $D(\phi(\tau)) \to U(\phi(\tau))D(\phi(\tau))U(\phi(\tau))^{-1}$ induces a nonlinear change of function in the world-sheet action. Despite this fact, the proposition shows that any topological B-brane is isomorphic with a direct sum of topological minimal branes. This is a consequence of the existence of the Smith normal form, which itself follows from the fact that our model's target space is $\mathbb{C}$.

Our category-theoretic discussion can be summarized by the following sequence of operations:

$$\begin{array}{ccccc} \mathrm{DG}_W & \stackrel{\text{restrict to cocycles}}{\longrightarrow} & \mathrm{P}_W & \stackrel{\text{divide by homotopies}}{\longrightarrow} & \mathrm{DB}_W \\ & & \downarrow & & \uparrow \\ & & \mathrm{M}_W & \stackrel{\text{divide by homotopies}}{\longrightarrow} & \mathrm{DM}_W \end{array} \tag{2.36}$$

where the right vertical arrow denotes inclusion and the left vertical arrow means that we identify objects which differ by isomorphisms in P^0_W (this induces corresponding identifications of the morphism spaces).

In physical terms the horizontal arrows implement passage to the cohomology of the boundary BRST-operator, in order to obtain the physical spectrum of the associated B-type brane. The category M_W is the direct sum completion of the category of minimal branes, while DM_W is its 'derived category'. The latter coincides with DB_W up to isomorphisms.

2.4 The minimal brane stratification of the factorization locus

For fixed closed string moduli t_i and rank r, the factorization condition (2.7) will generally admit families of solutions. Let us denote the space of solutions by $\mathcal{S}_t$ and discuss some of its properties. Later on we will briefly consider the union $\mathcal{Z} = \sqcup_t \mathcal{S}_t$, which is obtained by allowing t to vary.

Let $\mathcal{M}$ denote the quotient of $\mathcal{S}_t$ by the action of the similarity transformation (2.26). We have the orbit decomposition:

$$\mathcal{S}_t = \sqcup_{(p_1 \ldots p_r)\in\mathcal{M}} \mathcal{O}_{(p_1\ldots p_r)} \, . \tag{2.37}$$

[9]This, of course, does not mean that each (M, D) is a direct sum, since isomorphic objects in P^0_W cannot be identified in general. By definition the objects of DG_W, P_W and DB_W are pairs (M, D) satisfying $D^2 = W\mathbb{1}$, and *not* isomorphism classes of such pairs.

It is clear from section 2.3 that $\mathcal{M}$ consists of monic tuples $(p_1 \dots p_r) \in \mathbb{C}[x]^r$, subject to the divisibility constraints:

$$p_1|p_2|\dots|p_r|W\,. \tag{2.38}$$

Thus:

$$\mathcal{M} = \left\{ (p_1 \dots p_r) \in \mathbb{C}[x]^r \;\middle|\; p_1|p_2|\dots|p_r|W \text{ and } \mathrm{lcoeff}(p_j) = 1 \text{ for all } j = 1 \dots r \right\}. \tag{2.39}$$

To make this more explicit, recall the factorization (2.2) of the bulk Landau-Ginzburg superpotential. Writing $p_j(x) = \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{m_{i,j}}$ leads to the following description of the set of orbits:

$$\mathcal{M} \equiv \{\hat{m} = (m_{ij}) \in Mat(s, r; \mathbb{Z}) \mid 0 \le m_{i1} \le m_{i2} \le \dots \le m_{ir} \le k_i \ \forall i = 1 \dots s\,\}\,. \tag{2.40}$$

In particular, we can index orbits by the integral matrix $\hat{m}$. The orbit $\mathcal{O}_{\hat{m}}$ has the form:

$$\begin{aligned}
\mathcal{O}_{\hat{m}} = \Bigg\{ D = &\begin{bmatrix} 0 & SE_cT^{-1} \\ TJ_cS^{-1} & 0 \end{bmatrix} \Bigg| \\
&J_c = \mathrm{diag}\left(\prod_{i=1}^{s} (x - \tilde{x}_i(t))^{m_{i1}}, \dots, \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{m_{ir}} \right), \\
&E_c = \mathrm{diag}\left(\prod_{i=1}^{s} (x - \tilde{x}_i(t))^{k_i - m_{i1}}, \dots, \prod_{i=1}^{s} (x - \tilde{x}_i(t))^{k_i - m_{ir}} \right), \\
&S, T \in \mathrm{GL}(\mathbb{C}[x], r) \Bigg\}\,.
\end{aligned} \tag{2.41}$$

It is clear that $\mathbb{M}_j := (\mathbb{C}[x] \underset{p_j}{\overset{q_j}{\rightleftarrows}} \mathbb{C}[x])$ is the minimal brane $\mathbb{M}_{\mathbf{m}^{(j)}}$, where $\mathbf{m}^{(j)} = (m_{1j} \dots m_{sj})$ is the j'th column of the integral matrix $\hat{m}$. Thus branes belonging to the stratum $\mathcal{O}_{\hat{m}}$ are isomorphic with the direct sum $\mathbb{M}_{\mathbf{m}^{(1)}} \oplus \dots \oplus \mathbb{M}_{\mathbf{m}^{(r)}}$. This gives the following:

Proposition 2. *The strata $\mathcal{O}_{\hat{m}}$ of the solution space $\mathcal{S}_t$ are characterized by different minimal brane decompositions.*

Hence the minimal content of a brane 'jumps' along its deformation space each time one crosses from one stratum to another.

3. Deformations of D-branes by boundary operators

3.1 A first look at linearized deformations and obstructions

Let us consider a minimal brane $(M, D) \in Ob\mathrm{DG}_W$ and focus for simplicity on its boundary deformations. A basis of the even and odd components of $H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$ can be chosen as in (2.22). This defines (Grassmann-valued) linear coordinates $(\xi_\alpha, \eta_\alpha)$ on $H_M = G \otimes_{\mathbb{C}} H_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$, allowing us to write an arbitrary boundary observable $\mathcal{B}$ in the form:

$$\mathcal{B} = \sum_\alpha \xi_\alpha \phi_\alpha + \sum_\alpha \eta_\alpha \psi_\alpha\,. \tag{3.1}$$

It is clear that ξ_α have the same $\mathbb{Z}_2$-degree as $\mathcal{B}$, while η_α have opposite degree. The observables $\mathcal{B}$ can be used to deform the world-sheet action. Because the latter is constructed from the tachyon condensate [11]–[16], the simplest deformations correspond to:

$$D \to D' := D + \sum_\alpha \xi_\alpha \phi_\alpha + \sum_\alpha \eta_\alpha \psi_\alpha \,, \tag{3.2}$$

with even η_α and odd ξ_α (since the observable $1_G \otimes D \equiv D$ is odd). In this expression, ψ_α and ϕ_α denote representative cocycles of the associated $\mathcal{D}$-cohomology classes. Noticing that D enters linearly[10] in the boundary action S_∂ of [15], we find that this variation amounts to the following deformation:

$$\delta S_\partial = \sum_\alpha \xi_\alpha \int_{\partial\Sigma} d\tau \; [G^b_{-1}, \phi_\alpha] + \sum_\alpha \eta_\alpha \int_{\partial\Sigma} d\tau \; \{G^b_{-1}, \psi_\alpha\} \,, \tag{3.3}$$

where G^b_{-1} are modes of the twisted boundary supercurrent and $\partial\Sigma$ is the boundary of the world-sheet. Notice that the insertion of G^b_{-1} flips the total $\mathbb{Z}_2$ degree, so odd basis elements are associated with the even deformation parameters η_α, while even basis elements are associated with the odd parameters ξ_α. As pointed out in [2], the odd deformation parameters ξ_α drop out of the effective potential in the boundary preserving sector.[11] Accordingly, we will focus on deformations induced by turning on η_α, which also have a simpler physical interpretation.

Since η_α enter linearly in (3.3), they are analogous to the bulk flat coordinates t_i appearing in (2.3). Thus one expects η_α to play a role similar to that known for t_i from the theory of Frobenius manifolds. While we shall not attempt to do this here, we mention that one can cast the results of [2] in terms of a certain non-commutative version of Frobenius manifolds, a fact which should be used to give an intrinsic characterization of such coordinates. For the purpose of the present paper, we shall view η_α simply as linear coordinates on the vector space $H^1_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))[1]$.

Deformations of type (3.2) are generally obstructed, since D' need not satisfy the condition $D'^2 = W\mathbb{1}$. Writing $D' = D + \delta D$, the integrability equation takes the Maurer-Cartan form:

$$(\delta D)^2 + [D, \delta D] = 0 \,. \tag{3.4}$$

This can be studied via methods of algebraic homotopy theory as in [1]. Some basic aspects of this were recently discussed in [44, 45] in the context of topological Landau-Ginzburg orbifolds.

[10] More precisely, the term in the exponent of the path-ordered exponential of [15] depends linearly on D. As explained in [16], it is possible to add a supplementary term K to the boundary action, which depends quadratically on D (such a term was also considered in [11, 12, 14] in special cases). While this term does not affect BRST invariance of the world-sheet partition function, it is required if one wishes to have invariance under both generators of the untwisted $N = 2$ algebra. Since $\mathcal{W}_{\mathrm{eff}}$ is a generating function for tree-level *topological* string amplitudes, we can restrict to the B-twisted model. In this case, the quadratic term K does not play any role, and can be ignored (as explained in [16], this term does not contribute to localized correlators). The situation is similar to that encountered in [42] for B-type sigma models.

[11] Keeping them would require working with non-commuting supercoordinates in order to prevent their cancellation in $\mathcal{W}_{\mathrm{eff}}$ due to graded symmetrization [2].

As explained in [1] and [2] in a more general context, the effect of obstructions is encoded by the effective potential $\mathcal{W}_{\rm eff}$, which in our case is defined on $H^1_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$ and coincides with the generating functional of tree-level open string amplitudes in the boundary sector described by the brane $\mathbb{M}$. Then the even part $\mathcal{S}_t^{\mathbb{M}}$ of the deformation space of $\mathbb{M}$ (i.e. the space of those solutions to (2.9) or (3.4) which are continuously connected to the reference solution D) can be *locally* described as the critical set of $\mathcal{W}_{\rm eff}$:

$$\mathcal{S}_t^{\mathbb{M}} \approx_{\rm locally} \{\eta|\partial_{\eta_\alpha}\mathcal{W}_{\rm eff}=0 \text{ for all } \alpha\}\,. \tag{3.5}$$

This gives a local realization of $\mathcal{S}_t^{\mathbb{M}}$ as an affine variety inside $H^1_{\mathcal{D}}(\mathrm{End}_{\mathbb{C}[x]}(M))$. Of course, the deformation space $\mathcal{S}_t^{\mathbb{M}}$ of $\mathbb{M}$ is a subset of the full solution space $\mathcal{S}_t$.

3.2 Deformations of minimal branes

It is clear from equations (2.16) that a minimal brane $\mathbb{M}_{\mathbf{m}}$ has a single boundary deformation parameter, namely the non-vanishing constant C. However, such branes have $l+1$ linearized deformations, associated with the basis (2.22) of boundary observables. Thus l linearized deformations must be obstructed due to the effective potential.

Let us fix some closed string moduli t_i, for which $\mathbb{M}_{\mathbf{m}}$ is specified by the solution $(J_0(x;t), E_0(x;t))$ of the factorization condition (2.6). In the basis (2.22), the linear parameterization of tachyon deformations takes the form:

$$J(x;t,\eta) = J_0(x;t) - \sum_{\alpha=0}^{l}\eta_\alpha\, x^\alpha\,, \qquad \ell = 0\ldots\left[\frac{k}{2}\right], \tag{3.6}$$

with $l+1=\deg(\gcd(E,F)) \leq \ell+1 = \deg J_0(x)$. As mentioned in the previous subsection, we turn on only the even deformation parameters η_α. Notice that (3.6) ignores the modulus provided by the constant C in (2.16), which can be taken trivially into account. Thus — once the effect of obstructions is implemented — we should find a reduced moduli space which consists of a single point.

Writing $J_0(x;t) = x^{\ell+1}+\sum_{\alpha=0}^{\ell} a_\alpha(t)x^\alpha$, we find:

$$J^{(\ell+1)}(x;u) = x^{\ell+1} - \sum_{\alpha=0}^{\ell} u_{\ell+1-\alpha}\, x^\alpha\,, \qquad \ell = 0\ldots\left[\frac{k}{2}\right], \tag{3.7}$$

where we introduced the shifted parameters:

$$\begin{aligned} u_{\ell+1-\alpha} &:= \eta_\alpha - a_\alpha(t) \qquad \text{for} \qquad \alpha = 0\ldots l \\ u_{\ell+1-\alpha} &:= -a_\alpha(t) \qquad \text{for} \qquad \alpha = l+1\ldots \ell\,, \end{aligned}$$

Concentrating on linearized deformations of $\mathbb{M}_{\mathbf{m}}$ means that one allows only $u_{\ell+1-l}\ldots u_{\ell+1}$ to vary. However, it is convenient to permit variations of all u_α, which amounts to simultaneously describing linearized deformations of all minimal branes $\mathbb{M}_{\mathbf{m}}$ with a fixed value of $\ell = -1+\sum_{i=1}^{s} m_i$. This is especially convenient since (3.7) has the same form one would encounter for linearized boundary deformations of a minimal brane at the minimal model point, so we can use this 'minimal model' form even though we are studying deformations

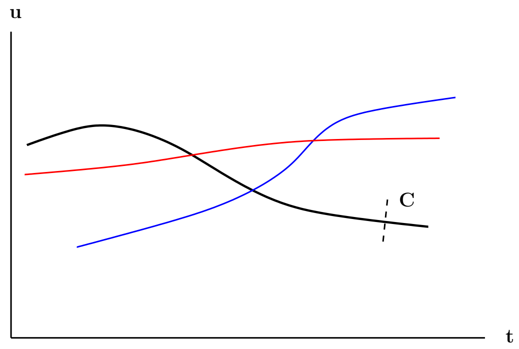

Figure 1: Schematic depiction of the 'total' joint deformation space $\mathcal{Z}^{(\ell+1)}$. Each branch is associated with a minimal brane, and carries a $\mathbb{C}^*$ bundle corresponding to the constant C in (2.16).

of minimal branes at a point in the closed string moduli space which is away from $t = 0$. Implementing obstructions to (3.7) will give a 'total' deformation space $\mathcal{S}_t^{(\ell+1)}$ consisting of a finite number of points, and one must use the supplementary information provided by $\mathbf{m}$ in order to identify the point associated with a given minimal brane.

For a linearized deformation (3.7), we can extend E to a non-polynomial function defined through:

$$E(x;t,u) = \frac{W(x;t)}{J(x;u)}. \tag{3.8}$$

Then the condition that E be a polynomial in x (i.e. $E_- = 0$, where the subscript denotes the singular part) imposes nonlinear constraints on u_α, which recover the 'total' deformation space:

$$\mathcal{S}_t^{(\ell+1)} := \{u \in \mathbb{C}^{\ell+1} | E_-(x;t,u) = 0\}. \tag{3.9}$$

This consists of a finite number of points, corresponding to a choice of $\ell+1$ zeroes of $W(x)$ in order to make up the polynomial $J^{(\ell+1)}(x)$; of course, each such choice corresponds to a given minimal brane $\mathbb{M}_{\mathbf{m}}$.

If we let the bulk moduli vary as well, we find the 'total' joint deformation space:

$$\mathcal{Z}^{(\ell+1)} = \{(t,u) \in \mathbb{C}^{\ell+1} \times \mathbb{C}^{k+1} | E_-(x;t,u) = 0\}, \tag{3.10}$$

which was computed for some examples in [12]. This affine algebraic variety is a branched multicover the affine space $\mathbb{C}^{k+1}$ of closed string moduli. The branching divisor coincides with the discriminant locus of $W(x)$. The joint deformation space of each $\mathbb{M}_{\mathbf{m}}$ (with $\sum_{i=1}^{s} m_i = \ell + 1$) is a $\mathbb{C}^*$-bundle over the corresponding branch of $\mathcal{Z}^{(\ell+1)}$ (figure 1).

3.3 Tachyon condensation and D-brane composites

We next consider deformations of a tachyon composite formed from a system of two branes $\mathbb{M}$ and $\mathbb{N}$. After explaining how the composite can be identified with an object of DG_W, we will extract the appropriate parameterization of its linearized deformations.

We start by discussing when a pair of odd morphisms in DG_W defines an allowed tachyon condensate, thus leading to a D-brane composite. Consider two objects $\mathbb{M}, \mathbb{N}$ and odd morphisms $f : \mathbb{M} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{M}$ in DG_W, with components $f_0 : M_+ \to N_-, f_1$:

$M_- \to N_+$ and $g_0 : N_+ \to M_-, g_1 : N_- \to M_+$. We want to know the conditions under which the maps f, g can be interpreted as (topological) tachyon vevs arising from strings stretching between the branes $\mathbb{M}, \mathbb{N}$. In that case, one obtains a D-brane composite "glued" by the tachyon vevs f, g and this physical interpretation requires that the morphism pair $\mathbb{M} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{N}$ be identified with a D-brane of our Landau-Ginzburg theory, i.e. an object of DG_W. To understand the identification, let us decompose the system as shown below:

$$\begin{tikzcd} \mathbb{M} \arrow[d, shift right, "f"'] \\ \mathbb{N} \arrow[u, shift right, "g"'] \end{tikzcd} \quad \cong \quad \begin{tikzcd} M_- \arrow[r, shift left, "E_M"] \arrow[dr, shift left, "f_1"] & M_+ \arrow[l, shift left, "J_M"] \arrow[dl, shift left, "f_0"'] \\ N_- \arrow[r, shift left, "E_N"] \arrow[ur, shift left, "g_1"] & N_+ \arrow[l, shift left, "J_N"] \arrow[ul, shift left, "g_0"'] \end{tikzcd} \tag{3.11}$$

Since (M_+, M_-) and (N_+, N_-) can be identified with $D2$ brane-antibrane pairs of the associated sigma model, it is clear that all eight maps on the right should be viewed as tachyon vevs of that model, arising from strings stretched between the sigma-model branes M_+, N_+ and their antibranes M_-, N_-. The net brane content of this system is given by the module $P_0 := M_+ \oplus N_+$, while the net antibrane content is $P_1 := M_- \oplus N_-$. Moreover, the net tachyon vevs are described by the odd morphisms $J : P_0 \to P_1$ and $E : P_1 \to P_0$ given by combining the relevant contributions:

$$J := \begin{bmatrix} J_M & g_0 \\ f_0 & J_N \end{bmatrix}, \qquad E := \begin{bmatrix} E_M & g_1 \\ f_1 & E_N \end{bmatrix}. \tag{3.12}$$

In these block matrices of morphisms, the column blocks are ordered by M_+, N_+ (in J) and M_-, N_- (in E), while the row blocks are ordered as M_-, N_- (in J) and M_+, N_+ (in E). The total tachyon condensate in the new object $\mathbb{P} = (P_0 \underset{J}{\overset{E}{\rightleftarrows}} P_1)$ is:

$$D = \begin{bmatrix} 0 & E \\ J & 0 \end{bmatrix}. \tag{3.13}$$

For the interpretation as a composite to hold, this must satisfy the condition $D^2 = W\mathbb{1} \Longleftrightarrow EJ = JE = W\mathbb{1}$, in which case $\mathbb{P}$ is an object of DG_W with which the morphism pair $\mathbb{M} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{N}$ should be identified. Using expressions (3.12), we find that this condition is equivalent with the following two systems of constraints:

$$\begin{aligned} J_M E_M + g_0 f_1 &= W\mathbb{1} \\ J_N E_N + f_0 g_1 &= W\mathbb{1} \\ J_M g_1 + g_0 E_N &= 0 \\ J_N f_1 + f_0 E_M &= 0 \end{aligned} \tag{3.14}$$

and:

$$\begin{aligned} E_M J_M + g_1 f_0 &= W\mathbb{1} \\ E_N J_N + f_1 g_0 &= W\mathbb{1} \\ E_M g_0 + g_1 J_N &= 0 \\ E_N f_0 + f_1 J_M &= 0\,. \end{aligned} \tag{3.15}$$

A pair $\mathbb{M} \overset{f}{\underset{g}{\rightleftarrows}} \mathbb{N}$ in DG_W which satisfies (3.14) and (3.15) will be called a *two-term generalized complex* (the terminology follows [17]). Our discussion shows that:
Two term generalized complexes over DG_W *can be identified with objects of* DG_W.

It can be shown that this identification is compatible with the category structure, namely such generalized complexes form a differential graded (dG) category which is equivalent with a full sub-category of DG_W.

Thus two-term generalized complexes describe D-brane composites obtained by tachyon condensation in a D-brane pair of the Landau-Ginzburg model. That condensation processes lead back to objects of the original D-brane category DG_W reflects the fact that this dG-category is quasiunitary in the sense of [17, 18] (namely, the original D-brane category is large enough to model the result of any D-brane composite formation process). In fact, the argument presented above is very similar to that given in [17] for the case of critical string theories. In particular, it is easy to extend this argument to generalized complexes built from an arbitrary number of objects of DG_W, and prove that the obvious differential graded category defined by such objects is equivalent with DG_W. We stress that this approach is both physically and mathematically fundamental and should precede the 'on-shell' approach based on the cone construction, as explained in detail in [17] and in the review [24]. The fundamental approach is off-shell since this includes *dynamical* information about the underlying theory. Mathematically, triangulated categories do not suffice, due to the the well-known lack of naturality of distinguished triangles. It is by now well-understood that such categories should be promoted to differential graded or A_∞ categories in order to avoid this problem. This, of course, has a clear physical meaning as explained in detail in [1] and [17]–[24]. The most clear-cut example is the case of A-models on Calabi-Yau manifolds, for which the derived category of the Fukaya category cannot even be defined within another approach.

We next discuss the spectrum and linearized deformations of the composite $\mathbb{M} \overset{f}{\underset{g}{\rightleftarrows}} \mathbb{N}$. Suppose that we are given maps f, g satisfying conditions (3.14) and (3.15). Then a variation $(\delta E_M, \delta J_M, \delta E_N, \delta J_N, \delta f, \delta g)$ agrees with these constraints if and only if the following Maurer-Cartan equation is satisfied:

$$(\delta D)^2 + [D, \delta D] = 0\,, \tag{3.16}$$

where:

$$\delta D = \begin{bmatrix} 0 & \delta E \\ \delta J & 0 \end{bmatrix} \tag{3.17}$$

is the induced variation of the tachyon condensate. The odd spectrum of the composite $P \equiv (\mathbb{M} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{N})$ is given by solutions of the linearized equation:

$$[D, \delta D] = 0\,, \tag{3.18}$$

modulo states of the form $[D, A]$ with even A — this, of course, recovers the odd cohomology $H^1_D(\mathrm{End}_{\mathbb{C}[x]}(P)) = H^1_D(\mathrm{End}_{\mathrm{DG}_W}(\mathbb{M} \oplus \mathbb{N}))$.

An important case, which will be relevant below, arises for $f, g = 0$, when the total tachyon condensate takes the form:

$$D = D_M \oplus D_N = \begin{bmatrix} 0 & 0 & E_M & 0 \\ 0 & 0 & 0 & E_N \\ J_M & 0 & 0 & 0 \\ 0 & J_N & 0 & 0 \end{bmatrix}\,, \tag{3.19}$$

which amounts to $E = E_M \oplus E_N$ and $J = J_M \oplus J_N$. This corresponds to starting with the direct sum object $\mathbb{M} \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{N} \equiv \mathbb{M} \oplus \mathbb{N}$. Then the spectrum of $\mathbb{P}$ is the direct sum $\oplus_{A,B=M,N} H_{D_{AB}}(\mathrm{Hom}_{\mathbb{C}[x]}(A, B))$, i.e. the total spectrum of boundary and boundary condition changing observables in the system of independent branes $\mathbb{M}$ and $\mathbb{N}$. This implies that we can parameterize linear deformations through:

$$\delta D = \sum_\alpha \eta^M_\alpha \psi^M_\alpha + \sum_\beta \eta^N_\beta \psi^N_\beta + \sum_\gamma \eta^{MN}_\gamma \psi^{MN}_\gamma + \sum_\gamma \eta^{NM}_\gamma \psi^{NM}_\gamma\,, \tag{3.20}$$

where η are even coordinates while ψ^{AB} form a basis of $H^1(\mathrm{Hom}(A, B))$, with $\psi^A := \psi^{AA}$. Since D describes the tachyon condensate of the brane $\mathbb{P} \in Ob\mathrm{DG}_W$, the arguments of subsection 2.1 imply that the associated variation of the boundary action is linear in all deformation parameters:

$$\begin{aligned} \delta S_\partial = & \sum_\alpha \eta^M_\alpha \int_{\partial\Sigma} dx\ \left\{G^{b,M}_{-1}, \psi^M_\alpha\right\} + \sum_\beta \eta^N_\beta \int_{\partial\Sigma} dx\ \left\{G^{b,N}_{-1}, \psi^N_\beta\right\} + \\ & + \sum_\gamma \eta^{MN}_\gamma \int_{\partial\Sigma} dx\ \left\{G^{b,MN}_{-1}, \psi^{MN}_\gamma\right\} + \sum_\gamma \eta^{NM}_\gamma \int_{\partial\Sigma} dx\ \left\{G^{b,NM}_{-1}, \psi^{NM}_\gamma\right\}. \end{aligned} \tag{3.21}$$

4. A closed form for the effective potential

4.1 The effective potential for minimal branes

In [2], we derived the consistency conditions for open-closed topological string amplitudes on the disk (namely the open string version of the WDVV equations, which includes the extension of the A_∞ relations of [46] to non-critical strings). As shown there, appropriately symmetrized open string correlators integrate to the generating function $\mathcal{W}_{\mathrm{eff}}$. In our case, this potential is defined on the total space $\mathbb{C}^{k+1} \times \mathbb{C}^{\ell+1}$ of joint linearized deformations, which plays the role of ambient space for the total joint deformation space $\mathcal{Z}^{(\ell+1)}$ of equation (3.10).

By generalizing results for minimal branes in various minimal models with low k and ℓ, the following closed expression for the effective potential was obtained in [2]:

$$\mathcal{W}_{\rm eff}(t,u) = \sum_{m=0}^{k+2} g_{k+2-m}(t) h^{(\ell)}_{m+1}(u)\,. \tag{4.1}$$

Here $g_{k+2-m}(t)$ are the coefficients of the bulk Landau-Ginzburg superpotential (2.1) and $h^{(\ell)}_{m+1}(u)$ are the symmetric polynomials defined through the expansion:

$$\log\left[1-\sum_{n=1}^{\ell+1} u_n y^n\right] := \sum_{m=1}^{\infty} h^{(\ell)}_m(u) y^m\,. \tag{4.2}$$

That (4.1) is indeed correct for all k,ℓ is a conjecture still awaiting proof, and we intend to address this in future work. In this paper, we shall accept that (4.1) is generally valid and investigate its consequences.

The deeper meaning of (4.1) is that correlators involving only boundary fields are completely determined by combinatorics. The property which underlies the appearance of symmetric functions is that all non-trivial correlators of odd[12] boundary observables have the same value. More precisely, they have the same non-zero value whenever the charge superselection rule is satisfied. With our normalization conventions, such nontrivial correlators are given by:

$$\left\langle \psi_{\alpha_1}\,\psi_{\alpha_2}\int G^-\psi_{\alpha_3}\cdots\int G^-\psi_{\alpha_{n-1}}\,\psi_{\alpha_n}\right\rangle = -\frac{1}{k+2}\,. \tag{4.3}$$

The $n-3$ integrations complicate the direct evaluation of these correlators due to the presence of contact terms, which is why we resorted to determining them by solving the consistency conditions.

It is possible to cast (4.1) into a more elegant form. For this, notice that the substitution $y = 1/x$ reduces (4.2) to:

$$\log J(x;u) = (l+1)\log x + \sum_{m=1}^{\infty} h_m\, x^{-m}\,, \tag{4.4}$$

where $J(x,u)$ is the boundary superpotential, parametrized as in (3.7) (the expansion in (4.4) is valid for large x). The effective potential (4.1) can thus be written as:

$$\mathcal{W}_{\rm eff}(t,u) = -\oint_{\mathcal{C}} \frac{dx}{2\pi i} W(x;t)\log J(x;u)\,, \tag{4.5}$$

where $\mathcal{C}$ is a closed counterclockwise contour encircling all $D0$-branes (i.e. all zeroes $x_i(u)$ of $J(x)$) and all cuts of the logarithm. Relation (4.5) is ambiguous due to the need of choosing appropriate branch cuts, but the ambiguity amounts to the freedom of adding an inessential constant to $\mathcal{W}_{\rm eff}$.

[12] While correlators involving even boundary observables are typically non-zero as well, the corresponding terms in the effective potential drop out upon (super-)symmetrization, so they do not play a role for our purpose. However, they must be taken into account when solving the consistency constraints [2].

From the interpretation of $\mathcal{W}_{\rm eff}$ as a deformation potential [2, 1], we expect that its u-critical set, defined by:[13]

$$\mathcal{Z}_{\rm crit} = \{(t,u)|\partial_u \mathcal{W}_{\rm eff}(t,u) = 0\} \tag{4.6}$$

should agree *locally* with the total joint deformation locus (3.10). More precisely, $\mathcal{Z}_{crit}$ should coincide with a branch of (3.10), provided that we restrict both to a small enough vicinity of a point (t,u) which lies on such a branch. Thus we are interested in polynomials $J(x;t,u)$ as in (3.6) which are close to a polynomial $J_0(x;t)$ associated with a minimal brane $\mathbb{M}_{\mathbf{m}}$ for which $\sum_{i=0}^{s} m_i = \ell + 1$ (in particular, $(\frac{W}{J_0})_- = 0$).

This expectation is in fact easy to check by writing $J(x;u) = \prod_{i=0}^{\ell}(x - x_i(u))$, which implies:

$$\partial_{u_\alpha} \mathcal{W}_{\rm eff}(t,u) = \oint_{\mathcal{C}} \frac{dx}{2\pi i}\left[W(x;t)\sum_{i=0}^{\ell}\frac{\partial_{u_\alpha} x_i(u)}{x - x_i(u)}\right] = \sum_{i=0}^{\ell} W(x_i(u);t)\partial_{u_\alpha} x_i(u)\,. \tag{4.7}$$

Thus the u-critical set of $\mathcal{W}_{\rm eff}$ is described by the linear system:

$$\sum_{i=0}^{\ell} \partial_{u_\alpha} x_i(u) W(x_i(u);t) = 0 \tag{4.8}$$

for the $\ell+1$ unknowns $W(x_i(u))$. Now notice that the $\ell+1$ parameters u in (3.7) suffice to specify the roots of the monic degree $\ell+1$ polynomial $J(x)$. As a consequence, the discriminant:

$$\Delta(u) := \det(\partial_{u_\alpha} x_i(u)) \tag{4.9}$$

is generically non-vanishing. Hence the only solution of (4.8) is $W(x_i(u)) = 0$ for all $i = 0\ldots\ell$. Thus each root of the polynomial $J(x)$ is also a root of $W(x)$. Since J is close to J_0, which divides W, the only possibility is that the multiplicities of the roots are smaller in $J(x)$ than in W. Thus J must divide W, and $\mathcal{Z}_{crit}$ must coincide with the J_0-branch of $\mathcal{Z}^{(l+1)}$ when restricted to a small enough vicinity of J_0.

Notice that this is a purely local statement. The variety $\mathcal{Z}_{crit}$ contains components associated with polynomials J that do not divide W. However, such components do not intersect the factorization locus $\mathcal{Z}^{(\ell+1)}$, so agreement is guaranteed in the vicinity of any true solution of the factorization problem (which, of course, is all that can be expected from the local analysis of [2, 1]).

Although the factorization $W = JE$ persists along the u-critical set $\mathcal{Z}_{crit}$, the cohomology in the boundary sector may change along this locus. In the remainder of this subsection, we discuss the condition on $\mathcal{W}_{\rm eff}(t,u)$ which ensures the preservation of a non-trivial spectrum. On this account we differentiate equation (4.7) a second time and obtain:

$$\partial_{u_\alpha}\partial_{u_\beta} \mathcal{W}_{\rm eff}(t,u) = \sum_{i=0}^{\ell} W(x_i(u);t)\ \left(\partial_{u_\alpha}\partial_{u_\beta} x_i\right) + \sum_{i=0}^{\ell} \partial_{x_i} W(x_i(u);t)\ \left(\partial_{u_\alpha} x_i \partial_{u_\beta} x_i\right). \tag{4.10}$$

[13] We treat bulk deformations as non-dynamical background fields, which is warranted at weak string coupling.

Suppose we stay at a point on the factorization locus, and we require, in addition, that $J|E$, i.e., that we are on the sub-locus:

$$\mathcal{Z}_{spec} := \{(t,u) \mid W = JE,\ J|E\} \subset \mathcal{Z}_{\text{crit}} \tag{4.11}$$

Since the boundary preserving spectrum is governed by the ideal $\mathcal{I}$ as defined in (2.20), this ensures that the number of odd (and even) cohomology classes takes the maximal value, $\ell + 1$. Note that $\mathcal{Z}_{spec}$ can equivalently be described by $\mathcal{Z}_{\text{spec}} = \{(t,u) \mid J|W,\ J|W'\}$. Therefore, we see from (4.10) that

$$\partial_{u_\alpha}\mathcal{W}_{\text{eff}}(t,u) = \partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}_{\text{eff}}(t,u) = 0 \quad \text{on } \mathcal{Z}_{\text{spec}}\,. \tag{4.12}$$

In order to show that (4.12) is true only on $\mathcal{Z}_{spec}$, we look at the vicinity of a point $(t_0,u_0) \in \mathcal{Z}_{spec}$, with J_0 and $E_0 = hJ_0$. Then, by the same line of argumentation as above, the non-vanishing discriminant $\Delta(u)$ ensures that

$$\mathcal{Z}_{spec} = \{(t,u) \mid \partial_{u_\alpha}\mathcal{W}_{\text{eff}}(t,u) = \partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}_{\text{eff}}(t,u) = 0\}\,. \tag{4.13}$$

An analogous argument can be made for the situation where J and E share a common factor G, whose degree is smaller than of J (cf. (2.18)). Then only a corresponding subset of the cohomology survives, and this reflected in a increased rank of $\partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}_{\text{eff}}$.

In physical terms this finding can be interpreted as follows: On the factorization locus $\mathcal{Z}_{\text{crit}}$ where $W = JE$, the boundary preserving parameters u_α do not have tadpoles and thus the theory has a stable, supersymmetric vacuum; however a non-trivial spectrum of boundary operators is not guaranteed. Only on the sub-locus $\mathcal{Z}_{\text{spec}} \subset \mathcal{Z}_{\text{crit}}$ one has a non-trivial spectrum, and this is reflected in zero eigenvalues of the 'mass-matrix' $\partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}_{\text{eff}}$.

4.2 The effective potential for general B-type branes

By solving the consistency conditions of [2] for correlators of various low-k models with several minimal branes, we checked that the equality of correlators (4.3) also holds for amplitudes involving odd boundary condition changing observables which mediate between such branes. Assuming that this property holds in general and accounting for the relevant combinatorics, we are lead to the following expression for the disk generating function of strings ending on arbitrary branes $\mathbb{M} = (M, D)$:

$$\mathcal{W}_{\text{eff}}(t,\eta) \ = -\oint_{\mathcal{C}} \frac{dx}{2\pi i} \log\bigl(\det J(x;\eta)\bigr) W(x;t)\,. \tag{4.14}$$

This amounts to replacing J by $\det J$ in (4.5).

The proof of local agreement of the critical locus of $\mathcal{W}_{\text{eff}}$ with the deformation space is similar to the case of minimal branes. Let us parameterize deformations of the $r \times r$ matrix J by $\eta = (\eta_\alpha)$ with $\alpha = 0 \dots H$, and write $\det J(x;\eta) = \prod_{i=0}^{L}(x - x_i(\eta))$ as a monic polynomial in x, where $L+1$ is the degree of $\det J(x)$. We also assume that $H \geq L$ and that the $H+1$ by $L+1$ matrix:

$$A(\eta) := (\partial_\alpha x_i(\eta))\,. \tag{4.15}$$

has maximal rank. Then:

$$\partial_{\eta_\alpha}\mathcal{W}_{\rm eff}(t,\eta)=\oint_{\mathcal{C}}\frac{dx}{2\pi i}\left[W(x;t)\sum_{i=0}^{L}\frac{\partial_{\eta_\alpha}x_i(\eta)}{x-x_i(\eta)}\right]=\sum_{i=0}^{L}W(x_i(\eta);t)\partial_{\eta_\alpha}x_i(\eta) \tag{4.16}$$

and we find that the η-critical locus $\mathcal{Z}_{\rm crit}$ of $\mathcal{W}_{\rm eff}$ is characterized by the condition that all roots of $\det J$ must also be roots of W. Since this is obviously the case along the joint deformation space $\mathcal{Z}$ (where $JE=W\mathbf{1}$ implies $\det J|W^r$), the inclusion $\mathcal{Z}\subset\mathcal{Z}_{\rm crit}$ is immediate. Local agreement of $\mathcal{Z}_{\rm crit}$ with $\mathcal{Z}$ after restriction to a sufficiently small vicinity of a point lying on $\mathcal{Z}$ follows by a simple continuity argument, as in the minimal case. We note that the inclusion $\mathcal{Z}\subset\mathcal{Z}_{\rm crit}$ also follows directly from (4.14), which implies:[14]

$$\partial_{\eta_\alpha}\mathcal{W}_{\rm eff}(t,\eta)=-\oint_{\mathcal{C}}\frac{dx}{2\pi i}\,{\rm Tr}[E(x;t,\eta)\partial_{\eta_\alpha}J(x;\eta)]\;=\oint_{\mathcal{C}}\frac{dx}{2\pi i}\,{\rm Tr}[J(x;\eta)\partial_{\eta_\alpha}E(x;t,\eta)]\,. \tag{4.17}$$

The right hand side of this identity vanishes along $\mathcal{Z}$, since by definition the matrix $E(x;t,\eta):=a\frac{W(x;t)}{J(x;\eta)}$ has no singular terms there.

Thus the boundary critical set of $\mathcal{W}_{\rm eff}$ agrees with the matrix factorization locus. This provides further evidence for our general ansatz (4.14).

4.3 Interpretation through holomorphic matrix models

Our proposal (4.14) for the effective potential admits a matrix model interpretation. For this, consider an antiderivative $V(x)$ of $W(x)$, i.e. a polynomial $V(x;t)$ in x (whose coefficients depend parametrically on t) which satisfies:

$$\partial_x V(x;t)=W(x;t) \tag{4.18}$$

(clearly V is defined only up to addition of a function $c(t)$ which is independent of x). Then integration by parts casts (4.5) into the form:

$$\mathcal{W}_{\rm eff}(t,\eta)=\oint_{\mathcal{C}}\frac{dx}{2\pi i}V(x;t)\sum_{i=0}^{L}\frac{1}{x-x_i(\eta)}=\sum_{i=0}^{L}V(x_i(\eta),t)\,,$$

where $\det J(x;\eta)=\prod_{i=0}^{L}(x-x_i(\eta))$ as before. Viewing the zeros $x_i(\eta)$ of $\det J(x;\eta)$ as eigenvalues of a complex $(L+1)\times(L+1)$ matrix $X(\eta)$, we can write the effective potential as:

$$\mathcal{W}_{\rm eff}(t,\eta)={\rm Tr}\;V(X(\eta),t)\,. \tag{4.19}$$

Thus $\mathcal{W}_{\rm eff}$ coincides with the classical action[15] of a holomorphic matrix model as defined and studied in [41] (the matrix model is holomorphic rather than hermitean because the eigenvalues $x_i(u)$ are complex). The zeroes $x_i(\eta)$ of $\det J$ can be viewed as the locations

[14] The sign change in the last equation reflects the fact that swapping J and E exchanges branes with antibranes.

[15] In this paper, we consider only the "small phase space". It would be interesting to extend the correspondence by coupling to topological gravity and including gravitational descendants. Presumably this involves the full dynamics of the holomorphic matrix model, rather than simply its classical action.

of $D0$-branes in the complex plane (=the target space of the Landau-Ginzburg model). Equation (4.19) shows that $\mathcal{W}_{\rm eff}$ is the 'potential energy' of this system of $D0$ branes when the latter is placed in the external 'complex potential' $V(x)$. Each critical configuration of this 0-brane system corresponds to a deformation of the underlying Landau-Ginzburg brane.

It has been known for a long time that the generalized Kontsevich model [37] is closely related to closed string minimal models coupled to topological gravity, but the open string version of this correspondence is not well understood. A link between certain topological D-branes and the auxiliary (Miwa) matrix of the Kontsevich model was proposed in [39], in the context of a non-compact Calabi-Yau realization of the underlying closed string model.

Our Landau-Ginzburg description gives a direct relation, which differs in spirit from that proposed in [39]: in the presence of several D-branes, the bulk Landau-Ginzburg field x is effectively promoted to a matrix $X(\eta)$. In [39], D-brane positions were mapped to the *auxiliary* matrix of the generalized Kontsevich model, so they parameterize backgrounds for the model's dynamics. In our case, D-brane positions are truly dynamical, being encoded by the matrix *variable* itself. The reason for this difference is that we study the D-brane potential (i.e. the generating function of scattering amplitudes for strings stretching between D-branes) rather than the flux superpotential (the contribution from RR flux couplings to the closed string sector) considered in [39].[16]

5. Examples of moduli spaces and tachyon condensation

In this section, we will apply our methods to analyze tachyon condensation in a few examples, for which we will explicitly determine the moduli spaces and their stratification. Some examples of boundary flows[17] in minimal models were previously discussed in [4]–[6].

Specifically, we study pure boundary deformations and switch off any bulk moduli; thus we shall set $W(x;t) = \frac{1}{k+2}x^{k+2}$. Starting with a system $\mathbb{M}_{\ell_1} \oplus \mathbb{M}_{\ell_2}$ of two independent minimal branes, we are interested in D-brane composite formation induced by turning on vevs for odd boundary preserving and boundary changing observables. This is the situation considered in subsection 3.3, with the choice $\mathbb{M} = \mathbb{M}_{\ell_1}$ and $\mathbb{N} = \mathbb{M}_{\ell_2}$. The effective potential (4.14) takes the form:

$$\mathcal{W}_{\rm eff}(u) = \frac{1}{k+2} \log \left[\frac{\det J(x;u)}{x^{\deg J(x)}}\right]\Big|_{x^{-k-3}}, \tag{5.1}$$

where we indicated that we take the coefficient of x^{-k-3} in the large x expansion of $\det J$ (notice that we divide out $x^{\deg J(x)}$ under the logarithm in order to have a well-defined Laurent expansion around $x = \infty$). Above, $J(x;u) \equiv J^{\ell_1,\ell_2}(x;u)$ is the 'total' map of

[16] Since D-branes carry RR charges, they can be viewed as backgrounds inducing a flux superpotential, which explains the different point of view used in [39]. Our interest, however, is in D-brane dynamics as dictated by tree-level scattering amplitudes of open strings.

[17] This is somewhat loose language, since, strictly speaking, there are no RG flows in the standard sense in topological models. More precisely, one can define a sort of RG flow at the level of string field theory, but such flows are homotopy equivalences with respect to the BRST operator so they always reduce to isomorphisms on-shell.

equation (3.12), which arises upon representing the composite $(\mathbb{M}_{\ell_1} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{M}_{\ell_2})$ as a single object $(P_0 \underset{J}{\overset{E}{\rightleftarrows}} P_1)$ of DG_W. Then relations (2.24) and (3.20) give the following explicit parameterization:

$$\begin{aligned} J^{\ell_1,\ell_2}(x;u) &= \begin{bmatrix} J^{(\ell_1+1)}(x;u^{[11]}) & g^{(\frac{1}{2}(\ell_1+\ell_2)+1)}(x;u^{[12]}) \\ f^{(\frac{1}{2}(\ell_1+\ell_2)+1)}(x;u^{[21]}) & J^{(\ell_2+1)}(x;u^{[22]}) \end{bmatrix} \\ &= \begin{bmatrix} x^{\ell_1+1} - \sum_{\alpha=0}^{\ell_1} u^{[11]}_{\ell_1+1-\alpha} x^\alpha & -\sum_{\gamma=0}^{\ell_{12}} u^{[12]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma} x^\gamma \\ -\sum_{\gamma=0}^{\ell_{21}} u^{[21]}_{\frac{1}{2}(\ell_1+\ell_2)+1-\gamma} x^\gamma & x^{\ell_2+1} - \sum_{\alpha=0}^{\ell_2} u^{[22]}_{\ell_2+1-\alpha} x^\alpha \end{bmatrix}. \end{aligned} \tag{5.2}$$

Upon substitution into (5.1), this agrees with the disk generating function found by solving the consistency constraints of [2]. Here $\ell_{12} = \ell_{21} = \min(\ell_1, \ell_2)$ and we traded the generic deformation parameters η for parameters u indexed in an manner which denotes their formal U(1) charges (cf. equation (2.25)). Moreover, we used superscripts to indicate the formal U(1) charges of J, f and g.

It is clear that the factorization locus in u-space (the locus where the factorization constraint (2.9) is satisfied) is determined by the equation

$$\det J(x;u) = x^{\ell_1+\ell_2+2}\,. \tag{5.3}$$

A simple way to satisfy this relation is to take all parameters u to vanish except for $u^{[12]}_\alpha$ (or $u^{[21]}_\alpha$), in which case the generic Smith normal form of J is $(1, x^{\ell_1+\ell_2+2})$ – where 'generic' means that $u^{[12]}_{\frac{1}{2}(\ell_1+\ell_2)+1} \neq 0$ (or $u^{[21]}_{\frac{1}{2}(\ell_1+\ell_2)+1} \neq 0$). This corresponds to the minimal brane content $\mathbb{M}_{-1} \oplus \mathbb{M}_{\ell_1+\ell_2+1}$, where $\mathbb{M}_{-1}$ is trivial. However, if we switch on only a single $u^{[12]}_{1+j}$ (or $u^{[21]}_{1+j}$), where $j \in \{\frac{|\ell_1-\ell_2|}{2} \dots \min(\frac{\ell_1+\ell_2}{2}, k - \frac{\ell_1+\ell_2}{2})\}$, the resulting Smith normal form corresponds to the process:

$$\mathbb{M}_{\ell_1} \oplus \mathbb{M}_{\ell_2} \xrightarrow{u^{[12]}_{1+j} \neq 0} \mathbb{M}_{\frac{1}{2}(\ell_1+\ell_2)+j+1} \oplus \mathbb{M}_{\frac{1}{2}(\ell_1+\ell_2)-j-1}\,. \tag{5.4}$$

This reproduces a result given in [6]. Notice that upon setting $\ell_2 = k - \ell_1$ and $j = k/2$, relation (5.4) includes brane-antibrane annihilation (in that case, the right hand side is the closed string vacuum since $\mathbb{M}_{k+1} = \mathbb{M}_{-1}[1]$ and $\mathbb{M}_{-1}$ is the trivial brane).

The complete deformation space is much more complicated since it is not restricted to purely block upper or lower off-diagonal deformations. This space is determined by condition (5.3), which amounts to a system of quadratic equations for the parameters u in $J(x;u)$. In general, this is a complicated affine algebraic variety, containing a multitude of strata associated with different Smith normal forms. The highest dimensional stratum corresponds to the Smith form $J = \mathrm{diag}(1, x^{\ell_1+\ell_2+1})$ and gives the minimal brane content $\mathbb{M}_{-1} \oplus \mathbb{M}_{\ell_1+\ell_2+1}$, as discussed above. Lower dimensional strata are obtained when the greatest common divisor $G_1(x)$ of the 1×1 minors of $J(x)$ is non-constant, and can be described by common factor conditions between the entries of $J(x)$. Because the general analysis of strata is rather complicated, it will not be presented here. Instead, we will illustrate it with a few examples further below.

Before doing so, let us mention that our considerations are not limited to two-brane systems. In fact, one can consider multi-brane configurations described by tachyon profiles $J^{\ell_1,\ldots,\ell_N}$, for example systems of N 'elementary' branes $\mathbb{M}_0$ described by:

$$J^{0,0,\ldots 0}(x;u) = \begin{pmatrix} x-u_1^{[11]} & -u_1^{[12]} & \ldots & -u_1^{[1N]} \\ -u_1^{[21]} & x-u_1^{[22]} & & \vdots \\ \vdots & & \ddots & \\ -u_1^{[N1]} & \ldots & & x-u_1^{[NN]} \end{pmatrix}. \tag{5.5}$$

We expect that arbitrary composites can be obtained by switching on suitable combinations of moduli. One may wonder what happens if $\sum \ell_i$ becomes arbitrarily large - clearly composites $\mathbb{M}_\ell$ with $\ell \geq k+2$ do not exist. In fact, the factorization condition (2.9) ensures that such branes cannot be formed; in other words, there cannot exist flat directions in $\mathcal{W}_{\rm eff}$ that would lead to such branes. However, configurations with an arbitrary number of allowed branes $\mathbb{M}_\ell$ with $\ell < k+2$ can be obtained.

As a specific example, consider a system of N identical branes $\mathbb{M}_\ell$, with $\ell \geq N-2$. One can check that switching on the tachyons $\{u_*\} \equiv \{u_{N-1}^{[1,N]}, u_{N-2}^{[2,N-1]} \ldots u_1^{[N/2,N/2+1]}\}$ (for N even) or $\{u_*\} \equiv \{u_{N-1}^{[1,N]}, u_{N-2}^{[2,N-1]} \ldots u_2^{[(N-1)/2,(N+1)/2+1]}\}$ (for N odd) produces a Smith form corresponding to the following deformation:

$$\Big(\mathbb{M}_\ell\Big)^{\oplus N} \overset{u_*\neq 0}{\longrightarrow} \bigoplus_{\ell'} C_{N+2,\ell}{}^{\ell'} \mathbb{M}_{\ell'} \,. \tag{5.6}$$

The quantities $C_{N+2,\ell}{}^{\ell'}$ are the SU(2) fusion rule coefficients at level k (with the understanding that $\ell = i-2$ labels an i-dimensional representation of SU(2)). This reproduces the boundary fusion rules found in [4], which are based on the coset construction of the $\mathcal{N}=2$ minimal models. It would be interesting to see whether there is a deeper relationship between matrix factorizations, Smith normal forms and boundary fusion rings — perhaps similar in spirit to [47].

After these general remarks, we now turn to the detailed analysis of a few examples. We will concentrate on deformations obtained by turning on tachyon vevs in systems of two minimal branes.

5.1 $\mathbb{M}_0 \oplus \mathbb{M}_{\ell-1} \to \mathbb{M}_{-1} \oplus \mathbb{M}_\ell$ (where $\mathbb{M}_{-1}$ is trivial)

This is the basic mechanism for recursively building up any D-brane $\mathbb{M}_\ell$ from the 'elementary' branes $\mathbb{M}_0$. We start with

$$J^{\ell_1=0,\ell_2=\ell-1} = \begin{bmatrix} x-u_1^{[11]} & -u_{\frac{1}{2}(\ell+1)}^{[12]} \\ -u_{\frac{1}{2}(\ell+1)}^{[21]} & x^\ell - \sum_{i=0}^{\ell-1} u_{\ell-i}^{[22]} x^i \end{bmatrix},$$

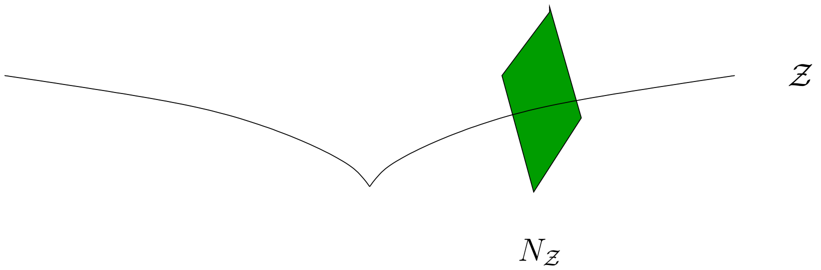

Figure 2: Schematic description of the moduli space for the composite of $\mathbb{M}_0$ and $\mathbb{M}_{\ell-1}$.

which depends linearly on $\ell+3$ complex parameters u. The determinant has the form:

$$\det J = x^{\ell+1} - \left(u_1^{[11]} + u_1^{[22]}\right) x^\ell + \sum_{i=1}^{\ell-1} \left(u_1^{[11]} u_{\ell-i}^{[22]} - u_{\ell+1-i}^{[22]}\right) x^i + u_1^{[11]} u_\ell^{[22]} - u_{\frac{\ell+1}{2}}^{[12]} u_{\frac{\ell+1}{2}}^{[21]}\,, \quad (5.7)$$

and gives a complicated expression for $\mathcal{W}_{\text{eff}}$. The critical locus $\mathcal{Z}_{\text{crit}}$ is characterized by the condition $\det J = x^{\ell+1}$, which gives the system of equations:

$$\begin{aligned} u_1^{[11]} + u_1^{[22]} &= 0 \\ u_1^{[11]} u_{\ell-i}^{[22]} - u_{\ell+1-i}^{[22]} &= 0\,, \quad \text{for } i = 1 \dots \ell-1 \\ u_1^{[11]} u_\ell^{[22]} - u_{\frac{\ell+1}{2}}^{[12]} u_{\frac{\ell+1}{2}}^{[21]} &= 0\,. \end{aligned} \quad (5.8)$$

This can be solved recursively in terms of $a := u_1^{[11]}$. From the first equation we find $u_1^{[22]} = -a$, while the $\ell-1$ conditions in the middle give the recursion relations:

$$u_{i+1}^{[22]} = a u_i^{[22]} \quad \text{for} \quad i = 1 \dots \ell-1\,, \quad (5.9)$$

with the solution:

$$u_i^{[22]} = -a^i \qquad \text{for} \qquad i = 1 \dots \ell\,. \quad (5.10)$$

Substituting in the final relation of (5.8), we obtain:

$$\mathcal{Z}\ : \qquad a^{\ell+1} + u_{\frac{\ell+1}{2}}^{[12]} u_{\frac{\ell+1}{2}}^{[21]} = 0\,. \quad (5.11)$$

Thus the factorization locus $\mathcal{Z}$ is the affine complex surface defined by equation (5.11) in $\mathbb{C}^3$, which is the well-known A_ℓ singularity. The singular point sits at the origin $u=0$ of the parameter space, as expected from the presence of obstructions to linearized deformations at that point. Equations (5.8) realize $\mathcal{Z}$ as a complete intersection in the original parameter space $\mathbb{C}^{\ell+3}$. When moving along $\mathcal{Z}$, one turns on vevs for tachyon fields between $\mathbb{M}_0$ and $\mathbb{M}_{\ell-1}$, thus forming a D-brane composite. Part of the virtual (i.e. linearized) deformations of this composite span the normal space to $\mathcal{Z}$ inside $\mathbb{C}^{\ell+3}$. Such normal deformations are of course obstructed, since the unobstructed directions are those tangent to $\mathcal{Z}$ (figure 2). To identify the strata and minimal brane decompositions, it is convenient to introduce the simplified notation:

$$u_1^{[11]} = a\,, \qquad u_{\frac{1}{2}(\ell+1)}^{[12]} = b\,, \qquad u_{\frac{1}{2}(\ell+1)}^{[21]} = c\,. \quad (5.12)$$

Then J takes the following form along the critical locus:

$$J = \begin{bmatrix} x-a & -b \\ -c & x^\ell - \sum_{i=0}^{\ell-1} a^{\ell-i}x^i \end{bmatrix}, \tag{5.13}$$

where the parameters are subject to the constraint:

$$bc + a^{\ell+1} = 0\,. \tag{5.14}$$

If $a \neq 0$, this equation shows that $b \neq 0$ and $c \neq 0$, which implies:

$$G_1 = \gcd\left(x-a, -b, -c, x^\ell - \sum_{i=0}^{\ell-1} a^{\ell-i}x^i\right) = 1\,. \tag{5.15}$$

Thus $p_1 = 1$, $p_2 = \det J = x^{\ell+1}$ and J can be brought to the form:

$$J \sim J_0 = \begin{bmatrix} 1 & 0 \\ 0 & x^{\ell+1} \end{bmatrix} \tag{5.16}$$

by a double similarity transformation. In this case, we find the minimal brane content $\mathbb{M}_{-1} \oplus \mathbb{M}_\ell$. The same situation occurs for $a = 0$ with $b \neq 0$ or $a \neq 0$.

At the origin $a = 0 = b = c = 0$ (the singular point of $\mathcal{Z}$), we find $G_1 = \gcd(x, 0, 0, x^\ell)$ $= x$, so $p_1 = x$, $p_2 = x^{\ell-1}$ and J can be brought to the form:

$$J \sim J_0 = \begin{bmatrix} x & 0 \\ 0 & x^\ell \end{bmatrix}. \tag{5.17}$$

This gives the minimal brane content $\mathbb{M}_0 \oplus \mathbb{M}_{\ell-1}$ which, as expected, is the original D-brane system.

We conclude that the deformation space $\mathcal{S} = \mathcal{Z}$ has two strata, characterized by the integer two-vector $m = (m_1, m_2)$:

$$\mathcal{S} = \mathcal{Z} = \mathcal{O}_{1,\ell} \sqcup \mathcal{O}_{0,\ell+1}\,. \tag{5.18}$$

Namely $\mathcal{O}_{1,\ell}$ is of the origin of the parameter space (the singular point of the A_1 singularity $\mathcal{Z}$), while $\mathcal{O}_{0,\ell+1}$ is the complement of $\mathcal{O}_{1,\ell}$ inside $\mathcal{Z}$. When moving away from the origin (even infinitesimally!), the minimal brane content jumps from $\mathbb{M}_0 \oplus \mathbb{M}_{\ell-1}$ to $\mathbb{M}_{-1} \oplus \mathbb{M}_\ell$.

To find the potential along the space $N_\mathcal{Z}$ normal to $\mathcal{Z}$, notice that normal directions can be described by variables $s_1 \dots s_{\ell+1}$ defined through the relations:

$$\begin{aligned} u_1^{[22]} &= s_1 - a\,, \\ u_2^{[22]} &= s_2 + s_1 u_1 - a^2, \\ &\vdots \\ u_\ell^{[22]} &= s_\ell + \sum_{j<\ell} s_j a^{i-\ell} - a^\ell\,, \end{aligned} \tag{5.19}$$

and:

$$u^{[12]}_{\frac{1}{2}(\ell+1)}u^{[21]}_{\frac{1}{2}(\ell+1)} = s_{\ell+1} + \sum_{j<\ell+1} s_j a^{\ell+1-j} - a^{\ell+1}\,.$$

With these substitutions, the effective potential (4.14) reduces to that of the brane $\mathbb{M}_\ell$, while a and the mode corresponding to the ratio $u^{[12]}_{\frac{1}{2}(\ell+1)}/u^{[21]}_{\frac{1}{2}(\ell+1)}$ decouple:

$$\mathcal{W}^{0,\ell-1}_{\rm eff}(u) \;\to\; \log\left[x^{\ell+1} - \sum_{i=0}^{\ell} s_{\ell+1-i}x^i\right]\Big|_{x^{-k-3}} = \mathcal{W}^{\ell}_{\rm eff}(s)\,.$$

As expected, the variables s_i can be identified with the 'special' deformation parameters of $\mathbb{M}_\ell$. The s_i-deformations are completely obstructed, as shown by their appearance in the effective potential, and by the fact that they spoil matrix factorization. This is in contrast with the flat directions tangent to $\mathcal{Z}$, for which the factorization condition is preserved. This can be checked by using the expressions:

$$J = \begin{bmatrix} x-a & -u^{[12]}_{\frac{1}{2}(\ell+1)} \\ -u^{[21]}_{\frac{1}{2}(\ell+1)} & x^{\ell-1}+\sum a^{\ell-1-i}x^i \end{bmatrix}, \tag{5.20}$$

$$E = x^{k+2-\ell}\begin{bmatrix} x^{\ell-1}+\sum a^{\ell-1-i}x^i & u^{[12]}_{\frac{1}{2}(\ell+1)} \\ u^{[21]}_{\frac{1}{2}(\ell+1)} & x-a \end{bmatrix}, \tag{5.21}$$

which satisfy $JE = EJ = x^{k+2}\mathbb{1}$ due to the constraint $u^{[12]}_{\frac{1}{2}(\ell+1)}u^{[21]}_{\frac{1}{2}(\ell+1)} = -a^{\ell+1}$.

5.2 $\mathbb{M}_1 \oplus \mathbb{M}_1 \to \mathbb{M}_2 \oplus \mathbb{M}_0$ or $\mathbb{M}_{-1} \oplus \mathbb{M}_3$

This example demonstrates how different non-trivial composites can be produced from one D-brane configuration, by appropriately tuning moduli. We consider the system $\mathbb{M}_1 \oplus \mathbb{M}_1$ with a tachyon condensate specified by:

$$J^{\ell_1=1,\ell_2=1} = \begin{bmatrix} x^2 - u^{[11]}_1 x - u^{[11]}_2 & -u^{[12]}_2 - u^{[12]}_1 x \\ -u^{[21]}_2 - u^{[21]}_1 x & x^2 - u^{[22]}_1 x - u^{[22]}_2 \end{bmatrix}.$$

To analyze the moduli space, we compute:

$$\det J := \det J = x^4 - (a+g)x^3 + (ag-ce-h-b)x^2 + (ah+bg-de-cf)x + bh - df\,, \tag{5.22}$$

where we introduced the simplified notation:

$$u^{[11]}_1 = a\,,\; u^{[11]}_2 = b\,,\; u^{[12]}_1 = c\,,\; u^{[12]}_2 = d\,,\; u^{[21]}_1 = e\,,\; u^{[21]}_2 = f\,,\; u^{[22]}_1 = g\,,\; u^{[22]}_2 = h\,. \tag{5.23}$$

The factorization locus is given by $\det J(x) = x^4$, which gives the equations:

$$\mathcal{Z}: \qquad a+g=0\,, \qquad ag-ce=h+b\,, \qquad bh=df\,, \qquad ah+bg=de+cf\,. \tag{5.24}$$

Hence $\mathcal{Z}$ is a four-dimensional affine variety, namely a complete intersection in $\mathbb{C}^8$. Since the first equation is linear, we can use it to eliminate the variable g in terms of a:

$$g = -a\,. \tag{5.25}$$

This reduces the remaining relations to:

$$\begin{aligned} h + b + a^2 &= -ce \\ bh &= df \\ cf + de &= a(h - b)\,, \end{aligned} \tag{5.26}$$

which present $\mathcal{Z}$ as a complete intersection in $\mathbb{C}^7$. If $a \neq 0$, we can eliminate b and h by using the first and third equation:

$$\begin{aligned} b &= -\frac{ace + a^3 + cf + de}{2a} \\ h &= -\frac{ace + a^3 - cf - de}{2a}\,. \end{aligned}$$

Then the second relation in (5.26) defines the following hypersurface in $\mathbb{C}^5$:

$$\mathcal{Z}_{\rm fact,0}\ : \qquad 4a^2 df = a^2(ce + a^2)^2 - (cf + de)^2\,. \tag{5.27}$$

Thus $\mathcal{Z}\backslash(a) = \mathcal{Z}_{\rm fact,0}\backslash(a)$, where (a) denotes the divisor $a = 0$. For $a = 0$, the system (5.26) becomes:

$$\begin{aligned} h + b &= -ce \\ bh &= df \\ cf + de &= 0\,. \end{aligned} \tag{5.28}$$

The first two conditions are the Viete relations for the polynomial $y^2 + cey + df$; thus h and b are the two solutions of the equation in y:

$$y^2 + cey + df = 0\,. \tag{5.29}$$

The remaining condition in (5.28) is the defining equation of the conifold (ODP) singularity in three dimensions. Hence the subvariety $\mathcal{Z}\cap(a)$ is a branched double cover of the conifold singularity.

Along the factorization locus, the matrix J takes the form:

$$J = \begin{bmatrix} x^2 - ax - b & -cx - d \\ -ex - f & x^2 + ax - h \end{bmatrix}, \tag{5.30}$$

whose parameters are subject to (5.26). Let us first assume that $ace \neq 0$. Then the greatest common denominator of all 1×1 minors of J is:

$$G_1 = \gcd\left(x^2 - ax - b, x^2 + ax - h, x + \frac{d}{c}, x + \frac{f}{e}\right) = \gcd\left(x^2 - ax - b, x - \frac{h - b}{2a}, x + \frac{d}{c}, x + \frac{f}{e}\right). \tag{5.31}$$

In the second equality, we performed a linear combination with constant coefficients of the first two polynomials. It is clear that G_1 is nontrivial (i.e. differs from a nonzero constant) if and only if the following conditions are satisfied:

$$\frac{h - b}{2a} = -\frac{d}{c} = -\frac{f}{e} = \alpha\,, \tag{5.32}$$

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with α a complex constant subject to:

$$\alpha^2 - a\alpha - b = 0\,. \tag{5.33}$$

Conditions (5.32) mean that all linear polynomials in the last form of (5.31) equal $x - \alpha$, while (5.33) is the requirement that $x - \alpha$ divides $x^2 - ax - b$. When these relations hold, we have $G_1 = x - \alpha$.

Equations (5.32) and (5.33) can be used to eliminate b, d, f and h:

$$b = \alpha^2 - a\alpha\,, \qquad d = -\alpha c\,, \qquad f = -\alpha e\,, \qquad h = \alpha^2 + a\alpha\,. \tag{5.34}$$

Substituting this into (5.26) gives the relations:

$$\begin{aligned} \alpha(a^2 + ce) &= 0 \\ a^2 + ce &= -2\alpha^2 \\ \alpha^2(\alpha^2 - a^2 - ce) &= 0\,, \end{aligned} \tag{5.35}$$

which are equivalent with $\alpha = a^2 + ce = 0$. Since α vanishes, equations (5.34) give $b = d = f = h = 0$. Thus the locus $\mathcal{C}_0$ of nontrivial G_1 is $b = d = f = h = a^2 + ce = 0$, a subvariety of $\mathcal{Z}$ which is isomorphic with an A_1 surface singularity. Along this locus, one finds $G_1 = x$ except at the origin, so that J can be brought to the form:

$$J \sim \begin{bmatrix} x & 0 \\ 0 & x^3 \end{bmatrix}. \tag{5.36}$$

This gives the minimal brane content $\mathbb{M}_0 \oplus \mathbb{M}_2$. By analyzing the case $ace = 0$, one finds that the only exception occurs at the origin of the parameter space, which is the singular point of $\mathcal{C}_0$. There one finds $G_1 = x^2$, with the minimal brane content $\mathbb{M}_1 \oplus \mathbb{M}_1$, which corresponds to the original (undeformed) D-brane system. On the complement $\mathcal{Z} \setminus \mathcal{C}_0$, one has $G_1 = 1$ and $J \sim \begin{bmatrix} 1 & 0 \\ 0 & x^4 \end{bmatrix}$, which gives the minimal brane content $\mathbb{M}_{-1} \oplus \mathbb{M}_3$. Hence the factorization locus has the stratification:

$$\mathcal{Z} = \mathcal{O}_{2,2} \sqcup \mathcal{O}_{1,3} \sqcup \mathcal{O}_{0,4}\,, \tag{5.37}$$

where:

$$\begin{aligned} \mathcal{O}_{2,2} &= \{0\} \;\; \text{(origin), with minimal brane content } \mathbb{M}_1 \oplus \mathbb{M}_1 \\ \mathcal{O}_{1,3} &= \mathcal{C}_0 \setminus \{0\}, \;\; \text{with minimal brane content } \mathbb{M}_0 \oplus \mathbb{M}_2 \\ \mathcal{O}_{0,4} &= \mathcal{Z} \setminus \mathcal{C}_0, \; \text{with minimal brane content } \mathbb{M}_{-1} \oplus \mathbb{M}_3\,. \end{aligned}$$

Deforming from the origin into the stratum $\mathcal{O}_{1,3}$ implements the process:

$$\mathbb{M}_1 \oplus \mathbb{M}_1 \longrightarrow \mathbb{M}_0 \oplus \mathbb{M}_2\,, \tag{5.38}$$

while deformations into $\mathcal{O}_{0,4}$ lead to:

$$\mathbb{M}_1 \oplus \mathbb{M}_1 \longrightarrow \mathbb{M}_{-1} \oplus \mathbb{M}_3\,. \tag{5.39}$$

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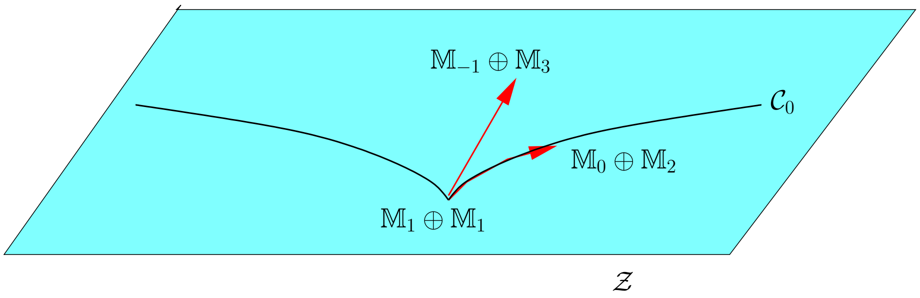

Figure 3: Realization of the processes $\mathbb{M}_1 \oplus \mathbb{M}_1 \longrightarrow \mathbb{M}_0 \oplus \mathbb{M}_2$ and $\mathbb{M}_1 \oplus \mathbb{M}_1 \longrightarrow \mathbb{M}_{-1} \oplus \mathbb{M}_3$ in the moduli space.

This is shown schematically in figure 3.

Let us focus on the stratum $\mathcal{O}_{1,3}$, noticing that its normal space in $\mathbb{C}^8$ can parameterized by complex quantities $\sigma_1, s_2^{[12]}, s_2^{[21]}, s_1 \dots s_3$ defined through:

$$\begin{aligned}
u_1^{[11]} &= \sigma_1 + \zeta_1 \,, \\
u_2^{[11]} &= -\sigma_1 \zeta_1 \,, \\
u_1^{[22]} &= s_1 - \zeta_1 \,, \\
u_2^{[22]} &= s_1 \zeta_1 \,, \\
u_1^{[12]} u_1^{[21]} &= s_2 - {\zeta_1}^2 \\
u_1^{[12]} u_2^{[21]} &= s_3 + s_1 {\zeta_1}^2 \\
u_2^{[12]} u_1^{[21]} &= -s_2 \sigma_1 + \sigma_1 {\zeta_1}^2 \\
u_2^{[12]} u_2^{[21]} &= s_2^{[12]} s_2^{[21]} - s_3 \sigma_1 - s_1 \sigma_1 {\zeta_1}^2 \,.
\end{aligned} \tag{5.40}$$

Performing this change of variables brings the effective potential to the form:

$$\begin{aligned}
\mathcal{W}_{\text{eff}}^{\ell_1=1,\ell_2=1} &\to \log \left[(1-\sigma_1)(x^3 - s_1 x^2 - s_2 x - s_3) - s_2^{[12]} s_2^{[21]} \right] \Big|_{x^{-k-3}} \\
&= \mathcal{W}_{\text{eff}}(\sigma, s)^{\ell_1=0,\ell_2=2} \,.
\end{aligned} \tag{5.41}$$

The variable ζ_1 and the ratios of the off-diagonal boundary changing moduli decouple, and represent flat directions along which composite formation occurs. Accordingly, matrix factorization persists along the locus $\mathcal{C}_0$, as can be checked from the expressions:

$$J^{\ell_1=0,\ell_2=2} = \begin{bmatrix} x^2 - x\,\zeta & -x\,u_1^{[12]} \\ -x\,u_1^{[21]} & x^2 + x\,\zeta \end{bmatrix}, \tag{5.42}$$

$$E^{\ell_1=0,\ell_2=2} = x^{k-2} \begin{bmatrix} x^2 + x\zeta & x\,u_1^{[12]} \\ x\,u_1^{[21]} & x^2 - x\zeta \end{bmatrix}, \tag{5.43}$$

by using the constraints given above.

6. Conclusions and outlook

We studied moduli spaces and tachyon condensation for D-branes in B-twisted minimal models of type A_{k+1} and their massive deformations. In particular, we showed that any D-brane in such models is isomorphic with a direct sum of 'minimal' rank one objects, which generalize the 'rational' branes known from the conformal point. This explains in what sense such branes play a distinguished role in minimal models. It is important to realize that the isomorphism relating a D-brane to a direct sum of minimal branes is *not* irrelevant and has a nontrivial physical realization on the world-sheet. In particular, the parameters of such isomorphisms are responsible for the fact that generic D-brane moduli spaces are algebraic varieties of positive dimension (as opposed to discrete collections of points). Therefore, it is *not* true that minimal branes exhaust the collection of boundary sectors in such models. On the contrary, the full D-brane category contains a continuous infinity of objects, while the minimal subcategory is finite. This distinction is physically meaningful and not a mathematical artifact.

We also showed that minimal brane decompositions induce a stratification of each D-brane's moduli space, where every stratum is associated with a different minimal brane content. Varying moduli inside a given stratum amounts to changing the isomorphism between the given D-brane and a fixed minimal brane decomposition, while crossing from a stratum to another amounts to changing the D-brane's minimal brane content. A combination of these processes implements transitions between different systems of independent minimal branes. As in [17, 21], our description is purely topological and should be supplemented with a stability condition, whose proper formulation in Landau-Ginzburg models is still unknown; we plan to return to this issue in future work.

A central point of the present paper is our proposal of a closed, synthetic expression for the effective tree-level potential $\mathcal{W}_{\rm eff}$ of open B-twisted strings ending on an arbitrary B-type brane. Since this quantity is the generating functional of open string scattering amplitudes, our generalized residue formula encodes the totality of such amplitudes for the B-twisted string. At the conformal point, this recovers all tree-level integrated CFT amplitudes containing only chiral primary insertions. Our approach presents $\mathcal{W}_{\rm eff}$ as a function of linearized deformation parameters, which play the physical role of 'special coordinates'. This goes beyond the mere computation of F-term equations through algebraic homotopy theory, which is ambiguous due to the freedom of choosing a minimal model of the associated differential graded algebra, an ambiguity which amounts to performing *power series* redefinitions of coordinates along the *formal* moduli space [1]. The geometric sense in which our coordinates are 'special' deserves further investigation, and we plan to report on this issue in future work. Another open issue under consideration is giving a direct derivation of the tree-level potential $\mathcal{W}_{\rm eff}$.

Our proposal for $\mathcal{W}_{\rm eff}$ admits a holomorphic matrix model description, which makes contact with the intuition that general branes in such models can be described as collections of D0-branes. More precisely, $\mathcal{W}_{\rm eff}$ arises as the *classical* potential of such a matrix model. It is natural to conjecture that the partition function of this model describes the coupling to topological gravity on the world-sheet. Establishing such a conjecture along the lines of [38] requires a detailed analysis of topological gravity on bordered Riemann surfaces, which has not yet been performed.

Another extension of the present work concerns D-branes in general B-twisted Landau-Ginzburg models [8], as formulated in [15, 16]. In the general case, D-brane deformations correspond to the moduli space of certain superconnections defined on the target space X of the model (which is a non-compact Calabi-Yau manifold), and one expects a much more complicated description. However, the residue-like proposal for $\mathcal{W}_{\rm eff}$ should generalize. One can expect substantial complications even for the simple case $X = \mathbb{C}^d$ with $d > 1$, since multivariate polynomial matrices do not generally admit a reduction to normal Smith form.[18]

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